A GRAVITY MODEL OF MORTALITY RATES FOR TWO POPULATIONS

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lifeMetrics

Motivation

- Most stochastic mortality models implicitly assume mortality rates are independent across populations
  - Potentially biologically unreasonable
  - Shared environmental factors, e.g., economic progress, cold winter or pandemic
- We propose a gravity model of the mortality rates of two interdependent populations

A gravity model

- Assume that dynamics of state variables (SVs) driving two related populations are attracted towards each other by a ‘gravitational’ pull dependent on relative pop sizes
- Case 1: 2 pops of equal size, e.g., males and females
  - Exert similar pulls on each other
  - Analogy of two equal-sized planets

A gravity model

- Case 2: large pop vs. small pop, e.g., national population vs. some subset
  - Large pop exerts pull on small pop, but small pop exerts negligible pull on large pop
  - Analogy is star vs planet or planet vs moon
  - This case is more relevant for potential hedgers, e.g., annuity provider considering an index hedge linked to a national mortality index
- We focus on this second case
- Pops are E&W males vs. CMI assured males

But there is a problem …

- Model depends on unobserved SVs
- SVs need to be estimated
- But we cannot estimate SVs without estimating params of SV processes, and we cannot estimate these without estimates of the SVs themselves
- A nice chicken and egg problem!
- See appendix for resolution

Model

- The gravity approach is illustrated in the context of M3B, a special case of the Age-Period-Cohort model
- This model is relatively tractable and has a cohort effect
- Can apply gravity approach to other models

1-pop M3B

- This model postulates

$$\log m_{t,x} = \beta_x + n^{-1}\kappa_t + n^{-1}\gamma_c$$

- *m* = death rate
- *Betas* are age-dependent SVs
- *kappas* are period SVs
- *gammas* are cohort SVs
- *x* = age, *t* = period, *c* = year of birth

1-pop M3B

- *Kappas* follow RW with drift

$$\kappa_t = \kappa_{t-1} + \mu + CZ_t$$

- First difference in *gammas* follow AR1

$$\Delta\gamma_c = \mu^{(\gamma)} - \alpha^{(\gamma)}\mu^{(\gamma)} + \alpha^{(\gamma)}\Delta\gamma_{c-1} + C^{(\gamma)}Z_c^{(\gamma)}$$

- SVs and parameters can be estimated using MLE

2-pop M3B

- Superscripts ‘(1)’ and ‘(2)’ refer to pops 1 and 2
- Pop 1 is large pop (E&W), pop 2 is small pop (CMI)
- Should NOT apply the 1-pop model to both pops
  - Nothing to stop kappa or gamma SVs of the two pops drifting apart
  - Violates notion of biological reasonableness
- Need to ‘connect’ the two pops dynamically

$\phi^{(\kappa)}$

2-pop M3B

- Postulate that *kappas* satisfy following RW:

$$\kappa_t^{(1)} = \kappa_{t-1}^{(1)} + \mu^{(1)} + C^{(11)} Z_t^{(1)} + C^{(12)} Z_t^{(2)}$$

$$\kappa_t^{(2)} = \kappa_{t-1}^{(2)} + \phi^{(\kappa)} (\kappa_{t-1}^{(1)} - \kappa_{t-1}^{(2)}) + \mu^{(2)} + C^{(21)} Z_t^{(1)} + C^{(22)} Z_t^{(2)}$$

- *phi_kappa* is a gravity parameter
- This pulls the pop-2 *kappas* towards the pop-1 *kappas*, but the latter are unaffected by the former
- Strength of pull depends on *phi_kappa*
- If *phi_kappa*=0, two populations are independent

2-pop M3B

- Postulate that *gammas* satisfy:

$$\begin{pmatrix} \gamma_c^{(1)} \\ \gamma_c^{(2)} \end{pmatrix} = \Phi_1 \begin{pmatrix} \gamma_{c-1}^{(1)} \\ \gamma_{c-1}^{(2)} \end{pmatrix} + \Phi_2 \begin{pmatrix} \gamma_{c-2}^{(1)} \\ \gamma_{c-2}^{(2)} \end{pmatrix} + \begin{pmatrix} \mu^{(\gamma 1)}(1-\alpha^{(\gamma 1)}) \\ \mu^{(\gamma 2)}(1-\alpha^{(\gamma 2)}) \end{pmatrix} + C^{(\gamma)} Z_c^{(\gamma)}$$

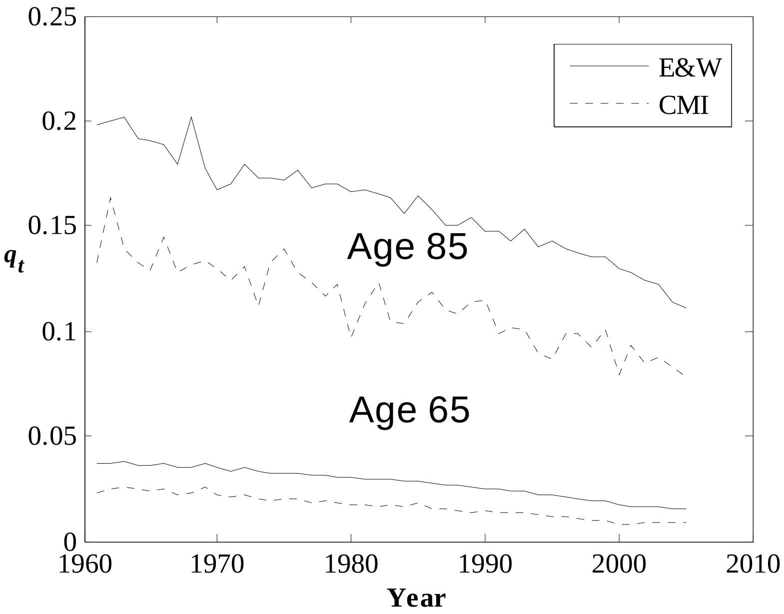
$$\Phi_1 = \begin{bmatrix} 1+\alpha^{(\gamma 1)}, 0 \\ \phi^{(\gamma)}, 1+\alpha^{(\gamma 2)} - \phi^{(\gamma)} \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} -\alpha^{(\gamma 1)}, 0 \\ 0, -\alpha^{(\gamma 2)} \end{bmatrix}$$

- Similar intuition

Historical *q* rates

- Historical *q* rates fall over time
- Lower for CMI than for E&W
- Higher and more volatile for higher ages
- Higher volatility for CMI than E&W

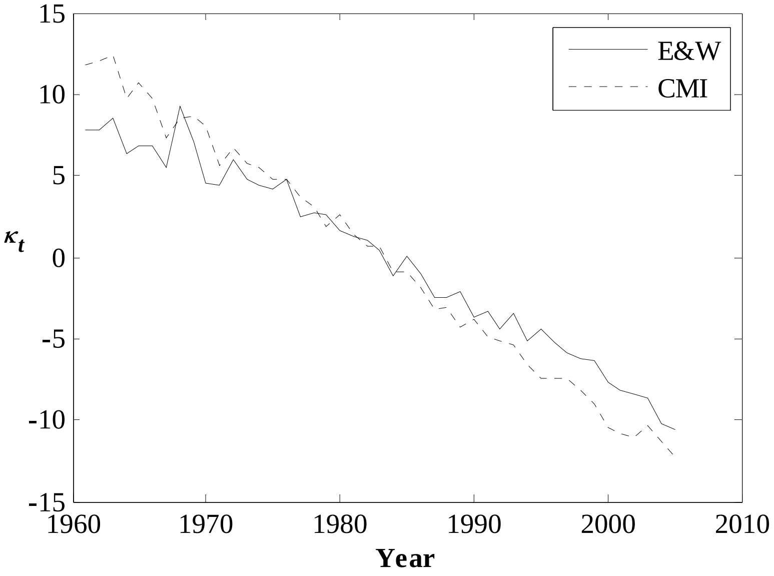


State variables: historical values

Kappas

Fall over time

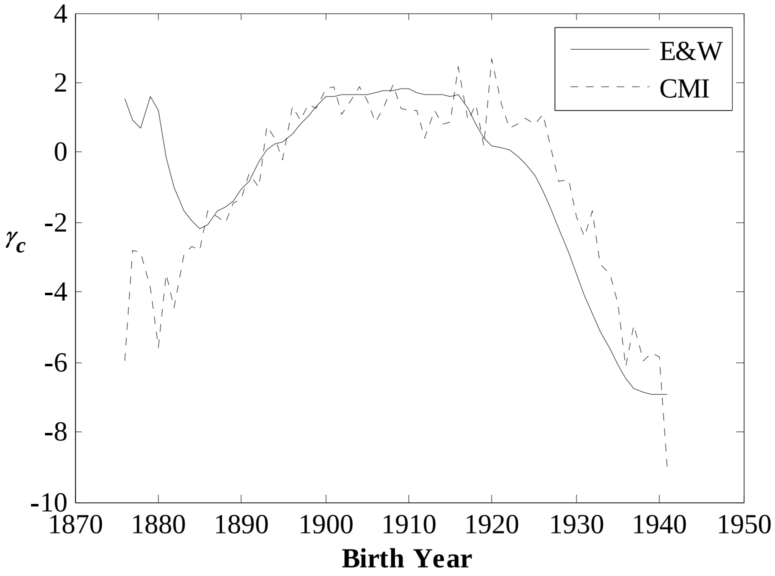
CMI fall at faster rate



Gammas

Broadly similar patterns

CMI more volatile



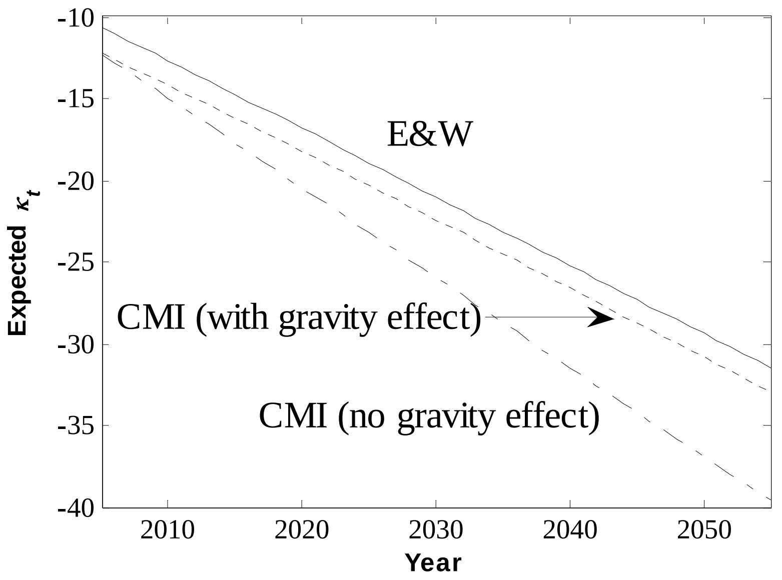
Expected future *kappas*

All projected to fall

Zero-gravity CMI falls faster and moves away from E&W

With-gravity CMI pushed towards E&W

Gravity effect apparent!



Significance of *phi_kappa*

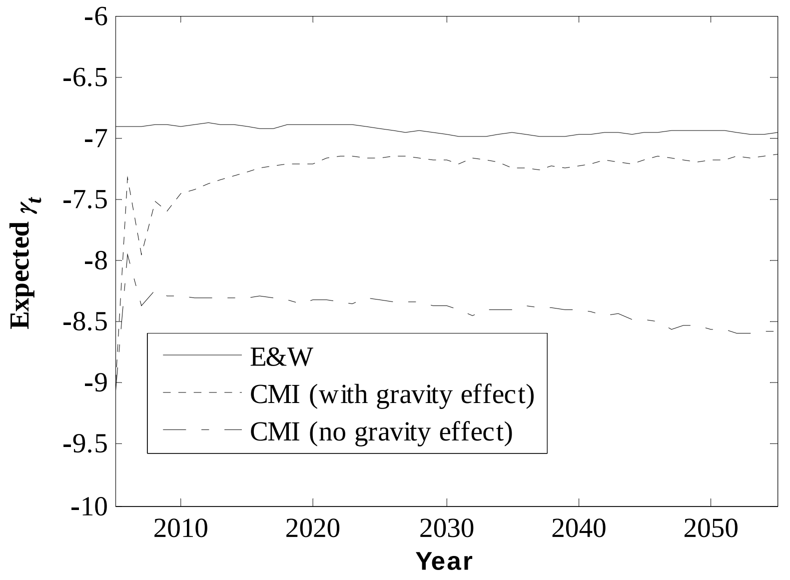
- MLE estimate of *phi_kappa* = 0.10505
- P-value (LR test) = 0.0105%
- Hence, gravity effect for *kappas* is highly significant
  - A 1-pop version of CMI would be misspecified
  - Need to take account of *kappa* gravity pull from large pop

Expected future *gammas*

E&W and zero-gravity CMI *gammas* have little trend

With-gravity CMI *gamma* pushed towards E&W *gamma*

Gravity effect again apparent!



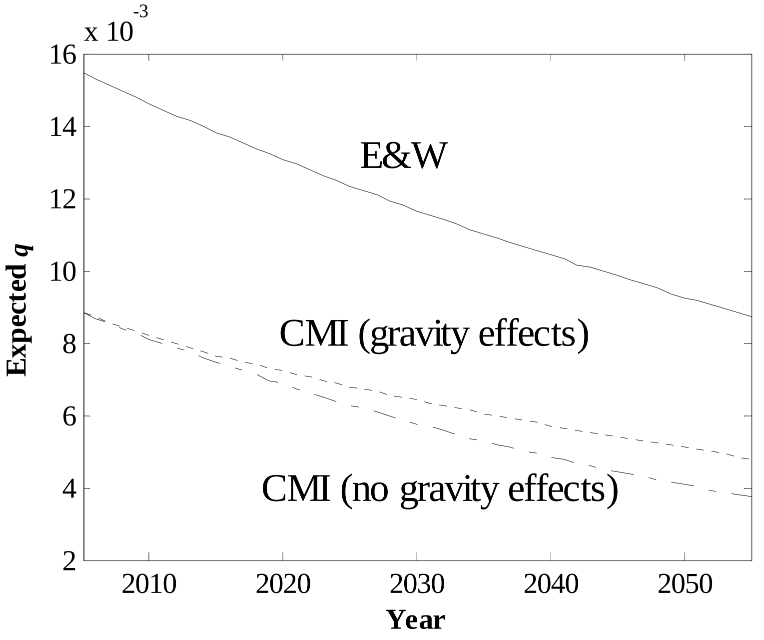
Significance of *phi_gamma*

- MLE estimate of *phi_gamma* = 0.1951
- P-value (*phi_gamma*=0) = 0.00002%
- Hence, gravity effects HIGHLY significant
- Again:
  - A 1-pop version of CMI would be misspecified
  - Need to take account of gravity pulls from larger pop

Expected future *q*

All expected *q*'s projected to fall

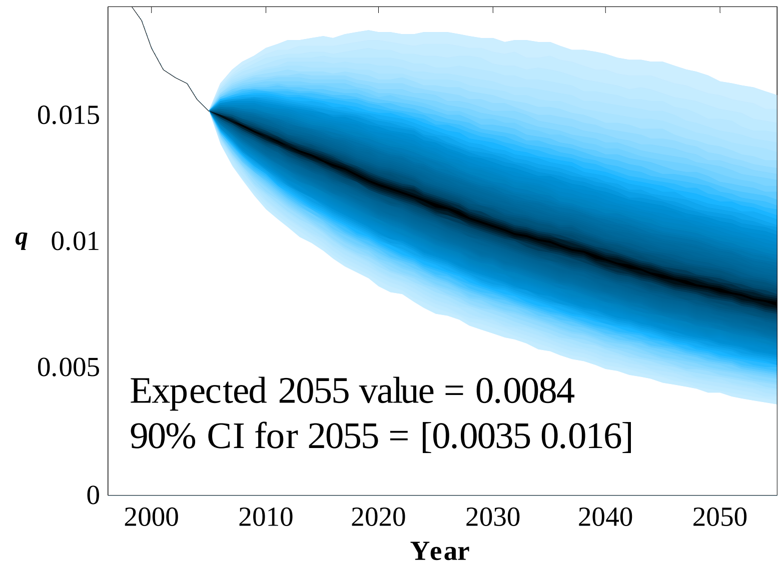
Impact of gravity effects to pull CMI *q* towards E&W *q* AND to moderate fall in CMI *q*



q fan chart projections: E&W, age 65

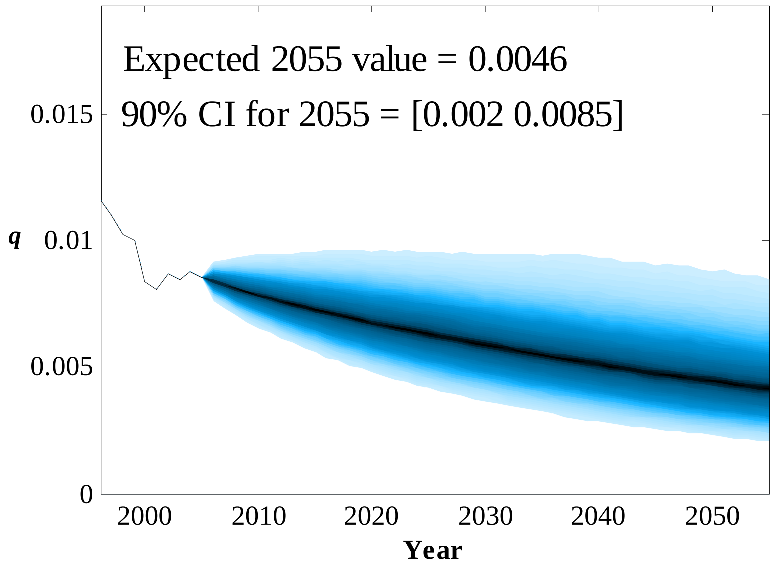
E&W *q*'s projected to fall but have wide fan charts

Future E&W *q* highly uncertain

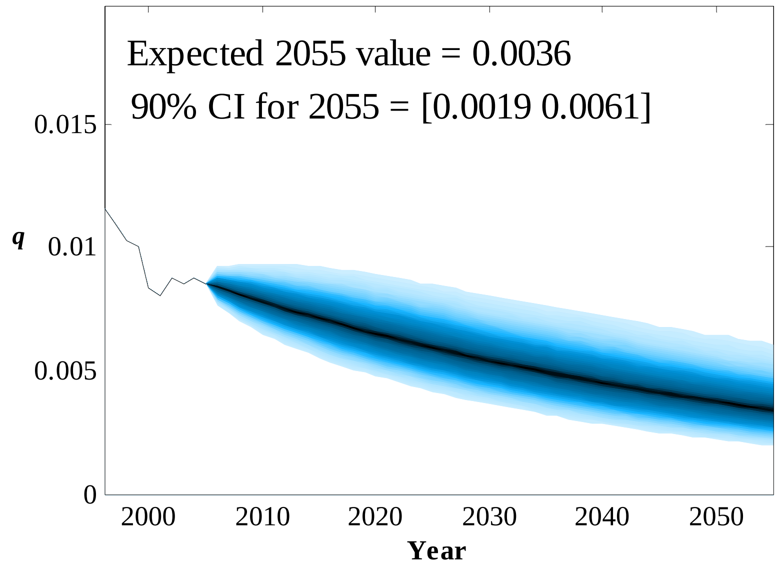


q fan chart projections: CMI, age 65

With gravity effects



Without gravity effects



Incorrectly ignoring gravity effects leads to over-estimation of mortality improvements **and** under-estimation of the uncertainty inherent in the projections

Application to term annuity prices

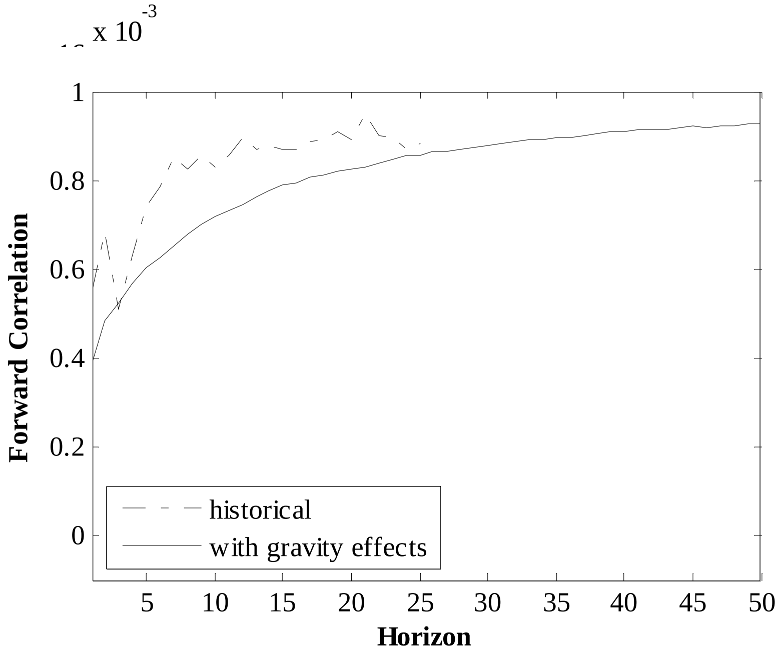
- Term annuity that pays $1 to survived age 90, age 65
- E&W TAP = 12.3012
- CMI (with gravity effects) = 13.4617
- CMI (no gravity effects) = 13.5758
- Gravity effects make a small difference to TAPs:
  – Less than the effect on q's

Forward mortality-improvement correlations

Model-based forward mortality-improvement correlations rise with horizon

Consistent with historical equivalents

Implications for hedging!



Conclusions: attractions of gravity model

- Gravity notion is plausible and intuitive
- All SVs and parameters can be estimated using MLE
- Approach is fast and easy to implement
- Approach can be used to:
  - Price mortality-related contracts
  - Model *q* vols or correlations
  - Implement index hedges

Conclusions: E&W vs CMI

- Results suggest that gravity (or 2-pop) effects are highly significant
  - Imply that CMI models that ignore gravity effects are misspecified
- Gravity effects make a considerable difference to projections of *q* rates and to forward correlations
- Results have important implications for risk management of mortality-dependent positions

Appendix: MLE procedure

- Remember our earlier chicken-and-egg problem?
- Can’t estimate SVs without estimates of their params, can’t estimate params without estimates of SVs
- We use following MLE procedure:
- Step 1: Estimate SVs using stand-alone or 1-pop model

Appendix: MLE procedure

- Step 2: Estimate params using SV estimates from stage 1
- [Step 1 + Step 2 = first estimation cycle]
- Step3: Re-estimate SVs using previous-stage estimates of params
- Step 4: Re-estimate params using previous-stage estimates of SVs
- [Step 3 + Step 4 = any subsequent estimation cycle]

Appendix: MLE procedure

- Repeat cycles as many times as required
- Need some cutoff point to determine when to stop
- We stop when estimates of SVs and estimates of params maximise likelihood
- ML is found after 3 cycles