

Yaari's LifeCycle Model in the 21st Century: Consumption Under a Stochastic Force of Mortality

(Joint work with H. Huang and T.S. Salisbury)

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- His model is at the foundation of much of modern micro-economics.
- The Yaari (1965) model was based on a deterministic force of mortality in which the entire survival curve is known at time zero.
- In this paper we extend the Yaari (1965) model – with no annuities – to a world with stochastic mortality rates.

The LifeCycle Model is Used to Provide Investment Advice

"...As far as I am aware, no one has challenged the view that if people were capable of it, they ought to plan their consumption, saving and retirement according to the principles enunciated by Modigliani and Brumberg in 1950s..."

Professor Angus S. Deaton, Princeton University, 2005

Force of Mortality

- Let $\lambda(t)$ denote the mortality rate of a cohort of a population. Let $\mathcal{F}_t = \sigma\{\lambda(q) \mid q \leq t\}$ be the filtration determined by λ . Then individuals in the population have lifetimes of length ζ satisfying

$$P(\zeta > s \mid \zeta > t, \mathcal{F}_\infty) = e^{-\int_t^s \lambda(q) dq}. \quad (1)$$

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- Assume further that $\lambda(t)$ is a Markov process, and define the survival function $p(t, s, \lambda)$ by

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- This gives the conditional probability of surviving from time t to time s , given knowledge of the mortality rate at time t . Therefore

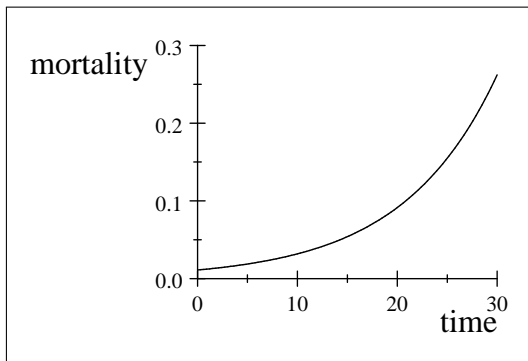
$$P(\zeta > s \mid \zeta > t, \mathcal{F}_t) = E \left[e^{-\int_t^s \lambda(q) dq} \mid \mathcal{F}_t \right] = p(t, s, \lambda(t)). \quad (3)$$

If $t = 0$ then we write $p(s, \lambda)$ for $p(0, s, \lambda)$.

Gompertz Mortality

A very popular law of mortality is the Gompertz law of mortality.

$$\lambda(t) = \frac{1}{b} \exp\left(\frac{x + t - m}{b}\right),$$



Notes: $x = 65$, $m = 89.3$, $b = 9.5$ and $p(0, 35, 0.0081) = 5\%$

The Yaari (1965) LifeCycle Model

- Objective Function:

$$J = \max_c E \left[\int_0^D e^{-\rho t} u(c(t)) \mathbf{1}_{\{t \leq \zeta\}} dt \right],$$

where $\zeta \leq D$ is the remaining lifetime satisfying $\Pr[\zeta > t] = p(t, \lambda_0)$.

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$$F_t(t) = v(t, F(t))F(t) + \pi_0 - c(t),$$

with $F(0) = W > 0$ and $F(D) = 0$.

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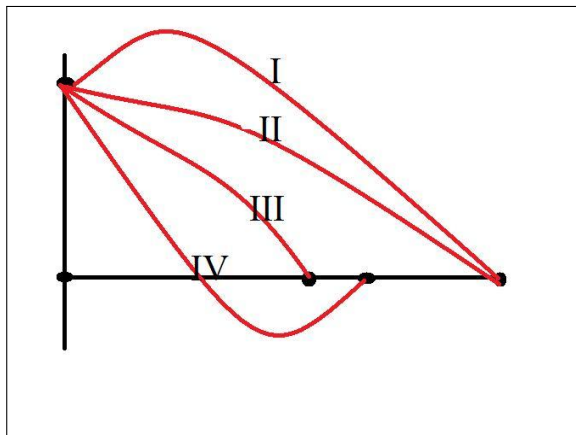
$$F_t(t) = v(t, F(t))F(t) + \pi_0 - c(t),$$

with $F(0) = W > 0$ and $F(D) = 0$.

- The investment return $v = v(t, F)$ is defined by:

$$v(t, F) = \begin{cases} r + \xi \lambda(t), & F \geq 0, \\ R + \lambda(t), & F < 0, \end{cases}$$

Graphical View of the Solution



Four Different Wealth Trajectories

Solution to LCM with DfM

- When $v(t) = r$, and $u(c) = c^{(1-\gamma)}/(1-\gamma)$ then by the Euler-Lagrange Theorem, the optimal $F(t)$ must satisfy a second-order non-homogenous differential equation in regions where $F(t) \neq 0$.

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- The PDE to solve is:

$$\begin{aligned} & F_{tt}(t) - \left(\frac{r - \rho - \lambda(t)}{\gamma} + r \right) F_t(t) + r \left(\frac{r - \rho - \lambda(t)}{\gamma} \right) F(t) \\ = & - \left(\frac{r - \rho - \lambda(t)}{\gamma} \right) \pi_0. \end{aligned}$$

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- Related literature: Leung (1990), Davies (1981), Lachance (2010).

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$$\begin{aligned} F(t) &= \left(F(0) - c^*(0) \int_0^t e^{ks} (p(s, \lambda_0))^{1/\gamma} e^{-rs} ds \right) e^{rt} \\ &= (F(0) - c^*(0) a_x^t(r - k, m^*, b)) e^{rt} \end{aligned}$$

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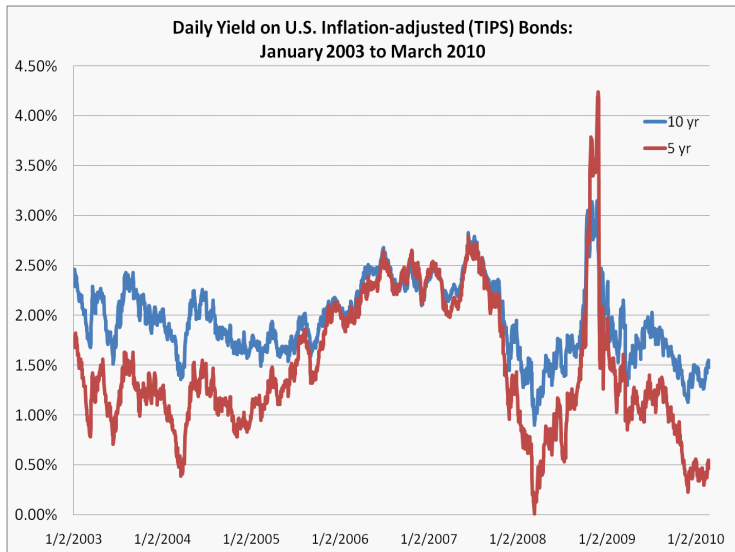
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- Initial consumption rate is...

$$c^*(0) = \frac{F(0)}{a_x^D(r - k, m^*, b)},$$

where $k = (r - \rho)/\gamma$ and $m^* = m + b \ln[\gamma]$.

Calibration: What Interest Rate Should we Use?



Numerical Results (DfM) #1

Optimal Consumption Rate				
Coefficient of Relative Risk Aversion (CRRA) $\gamma = 4$				
Nest Egg of \$100 Invested at Following REAL Rates...				
	$r = 0.5\%$	$r = 1.5\%$	$r = 2.5\%$	$r = 3.5\%$
Age 65				
5 Years Later				
10 Years Later				
20 Years Later				
30 Years Later				

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Nest Egg of \$100 Invested at Following REAL Rates...				
	$r = 0.5\%$	$r = 1.5\%$	$r = 2.5\%$	$r = 3.5\%$
Age 65	\$3.330	\$3.941	\$4.605	\$5.318
5 Years Later	\$3.286	\$3.888	\$4.544	\$5.247
10 Years Later	\$3.212	\$3.801	\$4.442	\$5.130
20 Years Later	\$2.898	\$3.429	\$4.007	\$4.627
30 Years Later	\$2.156	\$2.552	\$2.982	\$3.444

Numerical Results (DfM) #2

Optimal Initial Withdrawal Rate (IWR) from \$100

As a Function of Pension Income π_0

Depending on the Coefficient of Relative Risk Aversion

	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$	$\gamma = 8$
No Pension	6.330%	5.301%	4.605%	4.121%
$\pi_0 = \$1$				
$\pi_0 = \$2$				
$\pi_0 = \$5$				

Note: Gompertz Mortality ($m = 89.3$, $b = 9.5$) and $r = 2.5\%$

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Optimal Initial Withdrawal Rate (IWR) from \$100

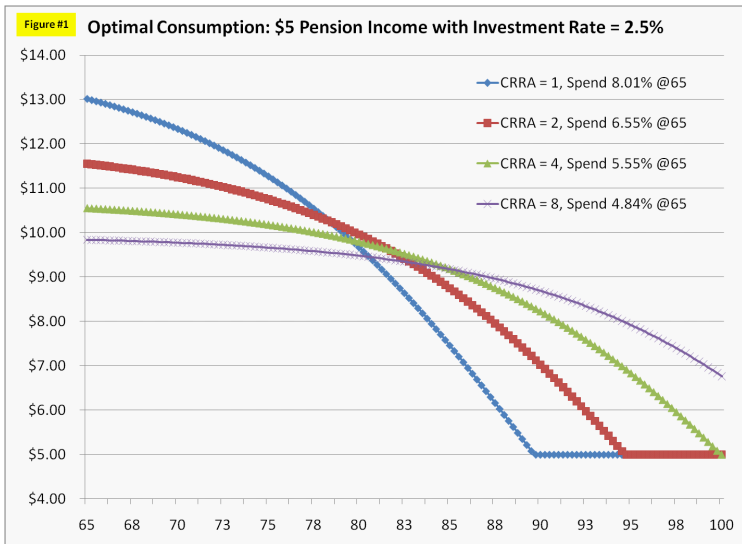
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	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$	$\gamma = 8$
No Pension	6.330%	5.301%	4.605%	4.121%
$\pi_0 = \$1$	6.798%	5.653%	4.873%	4.324%
$\pi_0 = \$2$	7.162%	5.924%	5.078%	4.480%
$\pi_0 = \$5$	8.015%	6.553%	5.551%	4.839%

Note: Gompertz Mortality ($m = 89.3$, $b = 9.5$) and $r = 2.5\%$

Retire with a \$100 Nest Egg and a \$5 per year pension...



Commercial Real Estate, Pages 22-25

New Advice to Retirees: Spend More at First, Cut Back Later

By HANA POLYAK

HOW much can you take out of your retirement nest egg each year without running out of money?

Not much, according to the standard, conservative advice of many financial planners. They often say that people who retire at the age of 65 can safely remove only about 4 percent of their portfolio each year, along with adjustments for inflation. On that basis, the annual withdrawal from a portfolio worth \$1 million would be just \$40,000.

But some experts have been making waves by suggesting that it may make more sense to withdraw higher amounts in the early years of retirement.

Ty Bernicke, a financial planner in Eau Claire, Wis., for example, says retirees generally spend less as they age, so that is reasonable for them to spend more when they are in retirement's early stages. Mr. Bernicke's conclusions, which relied on data from the Bureau of Labor Statistics' Consumer Expenditures Survey for 2002, were published as part in *The Journal of Financial Planning* (www.foxford.com/journal/articles/2005_10new_10b01a071.cfm).

Spending is practically every category, from housing to clothing to entertainment, declines with age, the data showed. The only category in which spending rises with age is health care, he said.

"It's almost a lag of war between inflation pushing costs up and human nature pulling them back down," Mr. Bernicke said.

People over 75 spend 26 percent less, on average, than those in the 65-to-74 age group. And the greater the age difference, the greater the percentage in spending. Those over 75 spend 40 percent less than those aged 65 to 69 and 50 percent less than those aged 60 to 64.

George and Kathy Magaw, both 68 and clients of Mr. Bernicke in Eau Claire, expect to spend less in the years ahead, Mr. Magaw said. He has decided to take early retirement from his job as a trading manager for a manufacturing company in June 2006, when he will be 63. Mrs. Magaw is not employed, the plan is spend time on his fishing boat and to go on waterfowl hunting trips in remote parts of Wisconsin. The couple also expect to visit grandchildren in Wisconsin and Connecticut.

"Am I going to end up spending a little bit more money up front?" Mr. Magaw said, but the period of higher expenditures should be relatively brief. It "won't be more than the first two or three years," he said.

Then he expects to reduce spending gradually on things like travel and entertainment — or making up for increases in health care, Mr. Magaw said.

The traditional advice that calls for an initial withdrawal of 4 percent is based on several assumptions. To compensate for inflation, the withdrawal rate would increase 1 percent every year. Someone with a \$1 million nest egg could take out \$40,000 the first year and \$42,000 the next year, for example.

And the nest egg would generally be invested at least 60 percent in stocks — as a further hedge against inflation — with the remainder in fixed-income investments and cash.

The approach is based on risk-assessment studies using all kinds of hypothetical examples of market returns. The withdrawal rates are intended to leave very little chance of running out of money.

"Our whole premise is that if you \$40,000 is to have the same purchasing power for the rest of your life, we have to adjust it," said Christine Fildes, senior financial adviser with F. Bruce Baker Associates, the

To Mr. Bernicke, a couple who spent \$40,000 over their first year of retirement may not need to spend as much when they are in their 80s. People who are 75 and older spend an average of \$174 a year on apparel and services, for example, while those who are 65 to 74 spend twice as much, based on the consumer survey he used. Those 75 and up spend an average of \$86 a year for entertainment, compared with \$177 for those 65 to 74.

According to his calculations, a couple in the first year of retirement at age 65, with expenditures of \$40,000, might be able to safely withdraw that much from a portfolio worth \$1 million — a 4 percent initial withdrawal rate. They would not run out of money so long as they reduced their spending later on according to the patterns shown in the survey, he said.

"Of course it depends on the mix of stocks and bonds in someone's portfolio," Mr. Bernicke said. "But a 4 percent withdrawal rate becomes very realistic."

He said that this rate could vary because of many factors, including a retiree's spending level and the size of the nest egg.

Others advocate lowering the pure savings in retirement, but for other reasons. In the October 2004 issue of *The Journal of Financial Planning*, Jonathan Guyton, a

financial planner in Minneapolis, advised an initial withdrawal rate as high as about 6 percent, drawing his conclusions from a study of market returns from 1870 to 2000. Mr. Guyton found that a person who retired in 1972, in the middle of a punishing bear market with very high inflation, could have reported a 6.2 percent initial withdrawal rate over 40 years with a portfolio that was 60 percent stocks. A portfolio with 65 percent in stocks could have borne a

Guyton said. "It's usually the last \$10,000 that puts the quality in 'quality of life.'"

In order not to take out more than 4 percent that first year, Mr. Guyton said, investors need to follow a few rules. To generate income, they must always sell winning stocks before losing stocks. They cannot add more than 4 percent a year to their withdrawal rate if inflation is higher than that. And no increases are permitted consecutively after a year of investment loss.

Mr. Guyton's research can be found at www.foxford.com/journal/articles/2004_10new_10b01a071.cfm.

The same, Mr. Bernicke's and Mr. Guyton's ideas have become more well-documented by many planners who worry that medical costs may rise so fast that they will erode a well-constructed financial plan.

The cost of prescription drugs, for example, has been rising more than three times as fast as inflation according to data from AARP, a lobbying organization for older Americans. Nursing home costs, meanwhile, have been rising 11 percent a year, according to survey by Metropolitan Life, an insurance company.

To help a agent those expenses, Mr. Bernicke said, many retirees try to buy insurance policies, but many just may not have enough to give more and withdraw less.

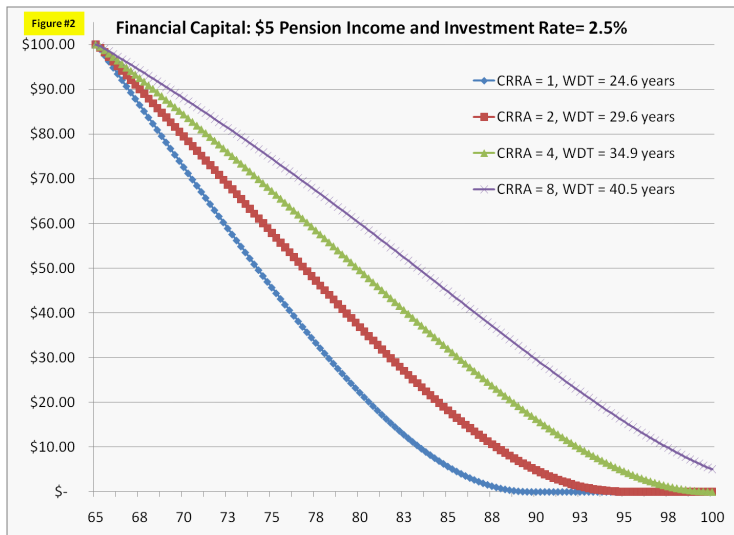
"I prefer a more conservative estimate of distribution," said Stephen Hancock of Hancock Wealth Advisory in Los Angeles. "I can't go back and say 'Oops. You shouldn't have been taking out all that money the lastest few years,' if someone doesn't have enough." He is also skeptical about the assumption that people will incur big costs on expenses as they shift into retirement. "You've have all this cash on your hands," he said.

"You've reduced actively spent money."

And Mr. Bernicke, who is 70 years old, said he is saving aggressively for his own retirement. Mr. Bernicke

Two planners challenge the traditional approach for managing that nest egg.

Wealth Trajectory



Numerical Results (DfM) #3

How Does Pensionization Impact Consumption?				
Percent	Risk Aversion $\gamma = 4$		Risk Aversion $\gamma = 8$	
Pensionized	Age 65	Age 80	Age 65	Age 80
0%				
20%				
40%				
60%				
100%				

Numerical Results (DfM) #3

How Does Pensionization Impact Consumption?				
Percent	Risk Aversion $\gamma = 4$		Risk Aversion $\gamma = 8$	
Pensionized	Age 65	Age 80	Age 65	Age 80
0%	\$4.605	\$4.007	\$4.121	\$3.844
20%	\$5.263	\$4.580	\$4.801	\$4.478
40%	\$5.795	\$5.042	\$5.385	\$5.024
60%	\$6.227	\$5.419	\$5.937	\$5.538
100%	\$6.330	\$6.330	\$6.330	\$6.330

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$$d\lambda(t) = \mu(t)\lambda(t)dt + \sigma\lambda(t)dB(t)$$

- How does optimal consumption behavior change and what is the impact of "longevity risk aversion" on the optimal plan?

Deterministic force of Mortality (DfM) World

$${}_{10}P_{90} = \frac{{}_{35}P_{65}}{{}_{25}P_{65}}$$

Conditional Survival Probability:

	$x = 65$	$x = 75$	$x = 85$	$x = 90$	$x = 95$	$x = 100$
To 65	1.000					
To 75	0.8659	1.000				
To 85	0.5733	0.6620	1.000			
To 90	0.3696	0.4268	0.6447	1.000		
To 95	0.1758	0.2031	0.3067	0.4757	1.000	
To 100	0.0500	0.0577	0.0872	0.1353	0.2844	1.000
λ_x	0.0081	0.0232	0.0667	0.1129	0.1911	0.3234

Stochastic force of Mortality (SfM) World

Conditional Survival Probability:

	$x = 65$	$x = 75$	$x = 85$	$x = 90$	$x = 95$	$x = 100$
To 65	1.000					
To 75	0.8659	1.000				
To 85	0.5733	?	1.000			
To 90	0.3696	?	?	1.000		
To 95	0.1758	?	?	?	1.000	
To 100	0.0500	?	?	?	?	1.000
λ_x	0.0081	?	?	?	?	?

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- $J(t, \lambda, F) =$

$$\max_{c(s) \text{ adapted}} E \left[\int_t^T e^{-\int_t^s (r + \lambda(q)) dq} u(c(s)) ds \mid \lambda(t) = \lambda, F(t) = F \right]$$

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$$e^{-\int_0^t (r+\lambda(q)) dq} J(t, \lambda(t), F(t)) + \int_0^t e^{-\int_t^s (r+\lambda(q)) dq} u(c(s)) ds.$$

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- By Ito's lemma, we have the following HJB equation for the value function:

$$\sup_c \{u(c) - cJ_F\} + J_t - (r + \lambda)J + rFJ_F + \mu(t)\lambda J_\lambda + \frac{\sigma^2 \lambda^2}{2} J_{\lambda\lambda} = 0$$

Solution under SfM: Part #2

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Solution under SfM: Part #2

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- Apply the 1st order condition $c^* = J_F^{-\frac{1}{\gamma}}$. We obtain

$$c^* = F a^{-\frac{1}{\gamma}}$$

and get the following equation for $a(t, \lambda)$:

$$a_t - (r\gamma + \lambda)a + \gamma a^{1-\frac{1}{\gamma}} + \mu(t)\lambda a_\lambda + \frac{\sigma^2 \lambda^2}{2} a_{\lambda\lambda} = 0$$

with boundary condition $a(T, \lambda) = 0$.

Numerical Results (SfM)

Main Question: How does the volatility of mortality (σ), impact the optimal initial withdrawal rate? The drift $\mu(t)$ of the mortality rate process is calibrated to fit a Gompertz survival curve ($m = 89.3, b = 9.5$), such that $p(35, 0.0081) = 5\%$

Optimal Initial Withdrawal Rate (IWR)

Volatility	$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$	$\gamma = 10$
$\sigma = 0$						
$\sigma = 15\%$						
$\sigma = 25\%$						

Notes: Retirement age 65, interest rate $r = 2\%$, mortality $\lambda_0 = 0.0081$

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Volatility	$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$	$\gamma = 10$
$\sigma = 0$	7.59%	6.12%	5.58%	5.02%	4.78%	4.61%
$\sigma = 15\%$	7.52%	6.12%	5.60%	5.04%	4.80%	4.62%
$\sigma = 25\%$	7.44%	6.12%	5.62%	5.06%	4.82%	4.63%

Notes: Retirement age 65, interest rate $r = 2\%$, mortality $\lambda_0 = 0.0081$

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- **THEOREM:** *Assume that the survival functions for the two models agree: $p^{\text{SfM}}(t, \lambda_0) = p^{\text{DfM}}(t, \lambda_0)$ for every $t \geq 0$, and that utility is CRRA(γ). There are three regimes: (a) $\gamma > 1 \implies c^{\text{SfM}}(0, \lambda_0, F) \geq c^{\text{DfM}}(0, F)$. (b) $\gamma = 1 \implies c^{\text{SfM}}(0, \lambda_0, F) = c^{\text{DfM}}(0, F)$. (c) $0 < \gamma < 1 \implies c^{\text{SfM}}(0, \lambda_0, F) \leq c^{\text{DfM}}(0, F)$.*

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- Proof in the paper...

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- Any horse race (i.e. comparison) between deterministic and stochastic mortality models, should ensure *rational mortality expectations*.
- Future research will examine the impact of annuities in such a model.