

# A Two-Population Mortality Model with Transitory Jump Effects

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Seventh International Longevity Risk and Capital Markets  
Solutions Conference

# Outline

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A Two-population Model without Jump Effects

Nonconcurrent Transitory Jumps

The Impact on Mortality Risk Securitization

Conclusion and Future Work

## Motivation

### Mortality jumps

- ▶ E.g. Spanish flu epidemic in 1918
- ▶ Important for mortality modeling and forecasting
- ▶ Important for pricing mortality-linked securities, especially those for hedging extreme mortality risk
- ▶ Modeling mortality jumps: Biffis (2005), Lin and Cox (2005), Chen and Cox (2009) and Cox et al. (2010)

## Motivation (cont'd)

### Multi-population models

- ▶ Model the potential correlations across different populations
- ▶ Ensure biologically reasonable mortality forecast
- ▶ Allow evaluating population basis risk
- ▶ Two-population mortality models: Carter and Lee (1992), Li and Lee (2005), Li and Hardy (2011), Dowd et al. (2011) and Cairns et al. (2011)

## Motivation (cont'd)

### Our Work

- ▶ Introduce jump effects to a two-population mortality model with a Lee-Carter structure
- ▶ Investigate the impact of mortality jumps on the securitization of mortality risk

## A Two-Population Model without Jump Effects

$$\ln(m_{x,t}^{(i)}) = \alpha_x^{(i)} + \beta_x^{(i)} \kappa_t^{(i)} \quad i = 1, 2$$

- ▶  $m_{x,t}^{(i)}$ : central death rate for population  $i$  at age  $x$  and in year  $t$
- ▶  $\kappa_t^{(i)}$ : period effect index for population  $i$  in year  $t$
- ▶  $\alpha_x^{(i)}$ : average level of mortality for population  $i$  at age  $x$
- ▶  $\beta_x^{(i)}$ : sensitivity to  $\kappa_t^{(i)}$  for population  $i$  at age  $x$

## Non-divergence

### Necessary conditions

1.  $\beta_x^{(1)} = \beta_x^{(2)}$  for all  $x$
2.  $\kappa_t^{(1)}$  and  $\kappa_t^{(2)}$  do not diverge over the long run

## Non-divergence (cont'd)

Modeling  $\kappa_t^{(1)}$  and  $\kappa_t^{(2)}$

$$\kappa_{t+1}^{(1)} = \kappa_t^{(1)} + \mu_{\kappa} + Z_{\kappa}(t+1)$$

$$\Delta_{\kappa}(t) = \kappa_t^{(1)} - \kappa_t^{(2)}$$

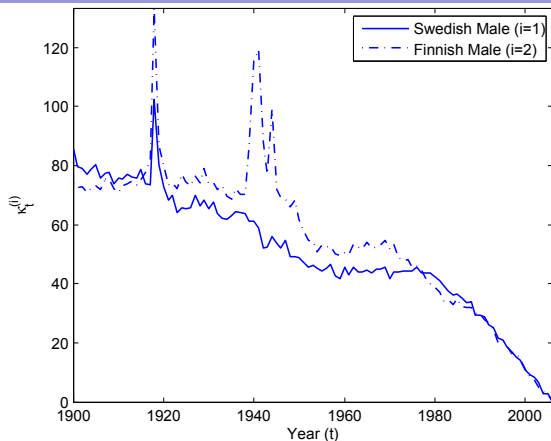
$$\Delta_{\kappa}(t+1) = \mu_{\Delta_{\kappa}} + \phi_{\Delta_{\kappa}} \Delta_{\kappa}(t) + Z_{\Delta_{\kappa}}(t+1)$$

- ▶  $|\phi_{\Delta_{\kappa}}| < 1$
- ▶  $(Z_{\kappa}(t), Z_{\Delta_{\kappa}}(t))' \sim \text{BVNorm}((0, 0)', V_Z)$

## Model fitting

### A two-stage approach

1. Estimate parameters  $\alpha_x^{(1)}$ ,  $\alpha_x^{(2)}$ ,  $\beta_x$ ,  $\kappa_t^{(1)}$  and  $\kappa_t^{(2)}$
2. Estimate the parameters in the time-series processes for  $\kappa_t^{(1)}$  and  $\Delta_\kappa(t)$



**Figure:** Estimates of the period effect indexes for Swedish male and Finnish male populations. (Data: sample period of 1900 to 2006 and sample age range of 25 to 84, obtained from Human Mortality Database (2011))

## Modeling nonconcurrent transitory jumps

$$\kappa_{t+1}^{(1)} = \hat{\kappa}_{t+1}^{(1)} + N_{t+1}^{(1)} Y_{t+1}^{(1)}$$

$$\kappa_{t+1}^{(2)} = \hat{\kappa}_{t+1}^{(2)} + N_{t+1}^{(2)} Y_{t+1}^{(2)}$$

$$\hat{\kappa}_{t+1}^{(1)} = \hat{\kappa}_t^{(1)} + \mu_{\kappa} + Z_{\kappa}(t+1)$$

$$\hat{\Delta}_{\kappa}(t) = \hat{\kappa}_t^{(1)} - \hat{\kappa}_t^{(2)}$$

$$\hat{\Delta}_{\kappa}(t+1) = \mu_{\Delta_{\kappa}} + \phi_{\Delta_{\kappa}} \hat{\Delta}_{\kappa}(t) + Z_{\Delta_{\kappa}}(t+1)$$

- ▶  $\{\hat{\kappa}_t^{(i)}\}$ : unobserved period effect index that is free of jumps
- ▶  $(Y_t^{(1)}, Y_t^{(2)})'$ : jump severities  $\sim \text{BVNorm}((\mu_Y^{(1)}, \mu_Y^{(2)}), V_Y)$
- ▶  $(Z_{\kappa}(t), Z_{\Delta_{\kappa}}(t))'$ : error terms  $\sim \text{BVNorm}((0, 0), V_Z)$

## Modeling nonconcurrent transitory jumps (cont'd)

- ▶  $N_t^{(i)}$ : jump count for population  $i$ 
  - ▶  $P(N_t^{(1)} = 1, N_t^{(2)} = 1) = p_1$
  - ▶  $P(N_t^{(1)} = 1, N_t^{(2)} = 0) = p_2$
  - ▶  $P(N_t^{(1)} = 0, N_t^{(2)} = 1) = p_3$
  - ▶  $P(N_t^{(1)} = 0, N_t^{(2)} = 0) = 1 - p_1 - p_2 - p_3$

## Parameter estimates

Parameters	Nonconcurrent-jump	No-jump
$\mu_{\kappa}$	-0.6372	-0.8057
$\mu_{\Delta_{\kappa}}$	-0.1269	-0.9803
$\phi_{\Delta_{\kappa}}$	0.9184	0.8486
$V_Z(1, 1)$	4.1752	17.3162
$V_Z(1, 2)$	1.0633	-11.0413
$V_Z(2, 2)$	2.7786	38.1830
$\mu_Y^{(1)}$	3.5824	N/A
$\mu_Y^{(2)}$	12.4430	N/A
$V_Y(1, 1)$	116.3952	N/A
$V_Y(1, 2)$	184.9353	N/A
$V_Y(2, 2)$	293.8356	N/A
$\rho_1$	0.0622	N/A
$\rho_2$	0	N/A
$\rho_3$	0.0496	N/A

## Likelihood ratio test

- ▶ Null model: no-jump model (log-likelihood  $l_1 = -634.1980$ )
- ▶ Alternative model: nonconcurrent-jump model (log-likelihood  $l_2 = -503.0726$ )
- ▶ Test statistics:  $2(l_2 - l_1) = 262.2508$
- ▶ Degree of freedom: 8
- ▶ P-value: 0

## An illustrative trade

- ▶ Agent A has life insurance liability  $L_t = 5 \sum_{x=25}^{44} q_{x,t}^{(2)}$ , where
 
$$q_{x,t}^{(2)} = 1 - e^{-m_{x,t}^{(2)}}$$
- ▶ Agent A issues mortality bond
  - ▶ 3 year maturity
  - ▶ a coupon at the end of each year at a rate of 4.5%
  - ▶ principal repayment linked to the index  $l_t = \frac{1}{20} \sum_{x=25}^{44} m_{x,t}^{(1)}$
  - ▶ principal repayment =  $\max\left(1 - \sum_{t=2007}^{2009} \text{loss}_t, 0\right)$
  - ▶  $\text{loss}_t = \frac{\max(l_t - 1.3l_{2006}, 0) - \max(l_t - 1.4l_{2006}, 0)}{0.1l_{2006}}$
- ▶ Agent B invests in mortality bond

## Pricing results

- ▶ Price the mortality bond by the economic pricing framework proposed by Zhou, Li and Tan (2010)

Model	Price	Quantity
No-jump	1.0178	0.2249
Nonconcurrent-jump	1.0087	0.1699

## Determinants of Supply and Demand

- ▶  $v_L$ : the accumulated values of the insurance liabilities
- ▶  $v_H$ : the accumulated values of the payouts from one unit of the bond

### Determinants

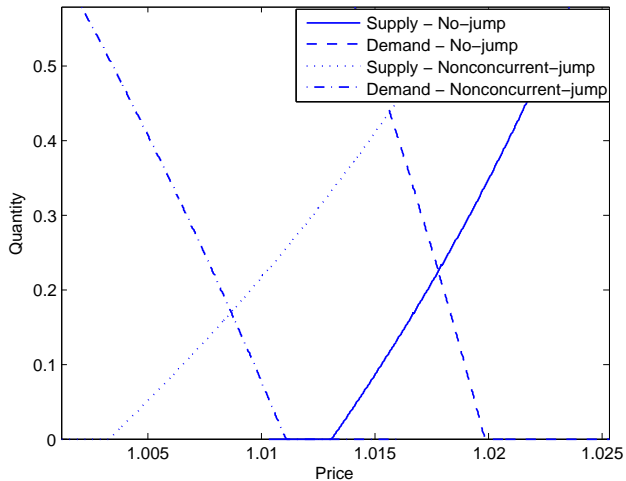
- ▶  $\mu_H$ : the expected value of  $v_H$
- ▶  $\sigma_H$ : the volatility of  $v_H$
- ▶  $\sigma_L$ : the volatility of  $v_L$
- ▶  $\rho$ : the correlation between  $v_L$  and  $v_H$

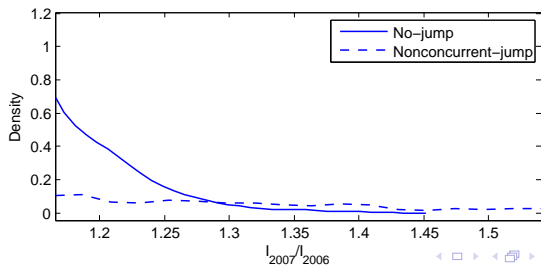
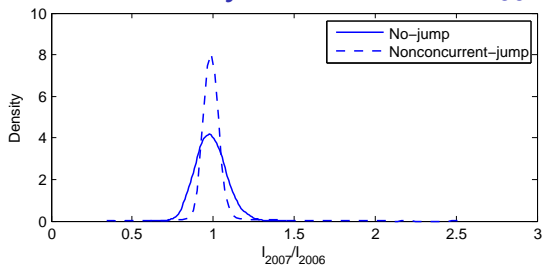
Estimates of  $\mu_H$ ,  $\sigma_H$ ,  $\sigma_L$  and  $\rho$ 

Model	$\mu_H$	$\sigma_H$	$\sigma_L$	$\rho$
No-jump	1.0198	0.1258	0.1102	-0.3829
Nonconcurrent-jump	1.0110	0.1573	0.0677	-0.5832

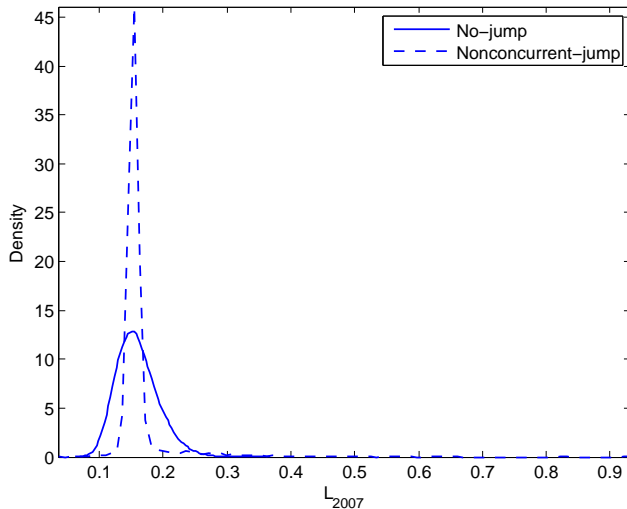
	$\mu_H \downarrow$	$\sigma_H \uparrow$	$\sigma_L \uparrow$	$ \rho  \uparrow$
Supply	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$
Demand	$\downarrow$	$\downarrow$	-	-

## The Supply and Demand Curves



Kernel smoothed density functions for  $I_{2007}/I_{2006}$ 

## Kernel smoothed density functions for $L_{2007}$



## Conclusion

- ▶ Incorporating nonconcurrent transitory jumps significantly improves the fit
- ▶ It also has significant impacts on the estimated mortality bond price

## Future work

- ▶ Measure population basis risk
- ▶ Allow permanent jumps

THANKS!