

# Measuring Basis Risk Involved in Longevity Hedges

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## Two Types of Mortality Derivatives

- Type I: Customized.
- E.g., the longevity swap agreed between Babcock International and Credit Suisse in 2009.
- Advantage: hedge can be perfect.
- Disadvantage: costly, poor liquidity.

## Two Types of Mortality Derivatives

- Type II: linked to a broad-based index.
- E.g., q-forwards, BNP/EIB bond.
- Advantage: cheaper, better liquidity.
- Disadvantage: basis risk.

# What Is Basis Risk?

- The risk associated with the difference in mortality experience between two populations.
- It exists due to, e.g., differing profiles of socioeconomic group, lifestyle and geography.
- We need models that help us measure the population basis risk involved in a longevity hedge.

# Objectives

- We need to model mortality of two populations.
- Most existing models are designed for a single population.
- Objectives of this study:
  - 1 to consider four models for modeling mortality of two populations;
  - 2 to quantify basis risk involved in a longevity hedge with these models.

# Independent Modeling

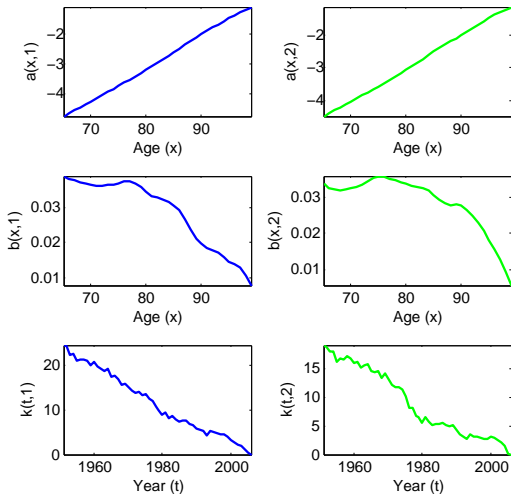
We model  $m(x, t, i)$ ,  $i = 1, 2$ , with:

$$\ln(m(x, t, i)) = a(x, i) + b(x, i)k(t, i) + \epsilon(x, t, i), \quad i = 1, 2,$$

where

- $a(x, i)$  is population  $i$ 's average mortality level at age  $x$ ,
- $k(t, i)$  represents the overall speed of mortality improvement for population  $i$ ,
- $b(x, i)$  indicates the sensitivity of  $\ln(m(x, t, i))$  to  $k(t, i)$ , and
- $\epsilon(x, t, i)$  is the error term.

# Independent Modeling



Estimates of parameters in the independent Lee-Carter model.

# Independent Modeling

- We model  $k(t, i)$  by an ARIMA(0,1,0) process, that is,

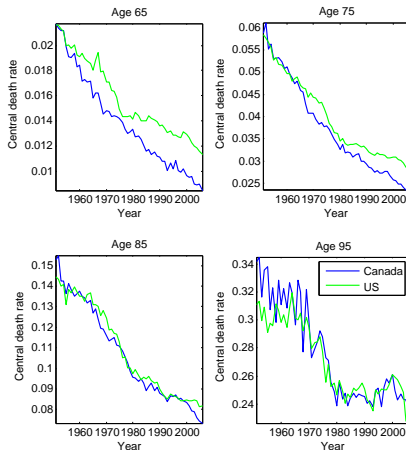
$$k(t, i) = c + k(t - 1, i) + \xi(t, i),$$

where  $c$  is the drift term and  $\{\xi(t, i)\}$  is a sequence of iid normal random variables.

- $\xi(t, 1)$  and  $\xi(t, 2)$  are independent.
- We can simulate future  $k(t, i)$  from the fitted processes.

# Independent Modeling

- Problem: dependence is ignored.



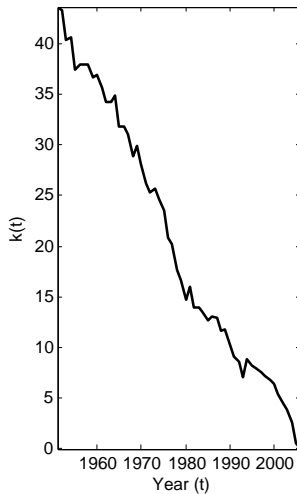
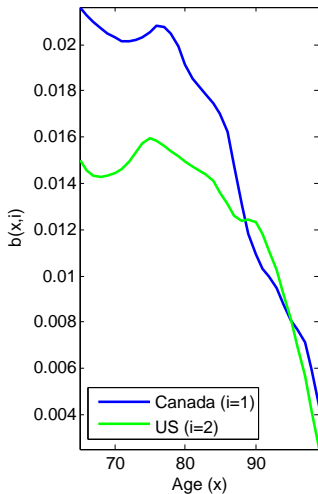
# The Joint-k Model

- Introduced by Carter and Lee (1992; Int'l J. of Forecasting).
- Assumes mortality rates of both populations are jointly driven by a single index:

$$\ln(m(x, t, i)) = a(x, i) + b(x, i)k(t) + \epsilon(x, t, i), \quad i = 1, 2.$$

- $k(t)$  is common to both populations.
- $k(t)$  is modeled by a random walk with drift.

# The Joint-k Model



Estimates of parameters in the joint-k model.

# The Joint-k Model

- Advantages:
  - 1 a single  $k$  is a parsimonious way of linking two mortality trajectories;
  - 2 greater consistency.
- Disadvantages:
  - 1 it implies mortality rates are perfectly correlated;
  - 2 may understate the actual level of basis risk

# A Co-integrated Lee-Carter Model

- Same as the independent model, we have:

$$\ln(m(x, t, i)) = a(x, i) + b(x, i)k(t, i) + \epsilon(x, t, i), \quad i = 1, 2.$$

- However, we treat  $\mathbf{k}(t) = (k(t, 1), k(t, 2))'$  as a vector:

$$\mathbf{k}(t) = \mathbf{c} + \mathbf{k}(t - 1) + \xi(t),$$

where  $\mathbf{c}$  is a vector of drift terms, and  $\{\xi(t)\}$  is a sequence of unserially uncorrelated bivariate normal random variables with mean zero and variance-covariance matrix  $\Sigma$ .

# A Co-integrated Lee-Carter Model

- The bivariate random walk gives same central projections as the independent random walks.
- But it yields narrower interval projections.
- The process for  $\mathbf{k}(t)$  has two unit roots (stochastic trends).
- Intuition: two driving forces governing the longevity improvements

# A Co-integrated Lee-Carter Model

- It is, however, possible that these two driving forces are the same.
- Econometricians call this situation co-integration.
- Cointegration refers to the situation when
  - 1  $\{k(t, 1)\}$  and  $\{k(t, 2)\}$  are both non-stationary, and
  - 2 there exists a constant  $\beta$  such that  $\{k(t, 1) - \beta k(t, 2)\}$  is stationary.
- We test cointegration with Engle and Granger's procedure.

# A Co-integrated Lee-Carter Model

- 1 Running the augmented Dickey-Fuller test on  $\{k(t, 1)\}$  and  $\{k(t, 2)\}$ . I.e., testing  $(H_0): \gamma = 0$  in

$$\Delta k(t, i) = \sum_{j=1}^p \Delta k(t-j, i) + \gamma k(t-1, i) + \xi(t),$$

- 2 Estimate  $\beta$  by linear regression:

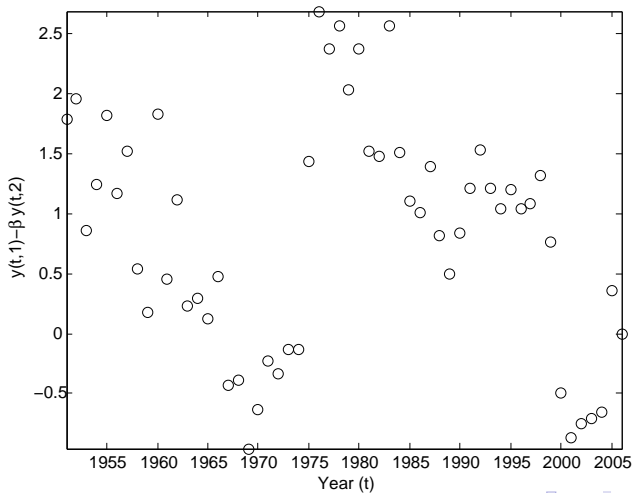
$$k(t, 1) = \alpha + \beta k(t, 2) + u(t).$$

- 3 Examine if  $\{u(t)\}$  is stationary by performing the augmented Dickey-Fuller test on  $\{u(t)\}$ . I.e., testing  $H_0 : \pi = 0$  in

$$\Delta \hat{u}(t) = \sum_{j=1}^p \Delta \hat{u}(t-j) + \pi \hat{u}(t-1) + e(t).$$

# A Co-integrated Lee-Carter Model

Conclusion:  $k(t, 1)$  and  $k(t, 2)$  are cointegrated.



# A Co-integrated Lee-Carter Model

- Phillips' (1991) triangular representation for cointegrated series:

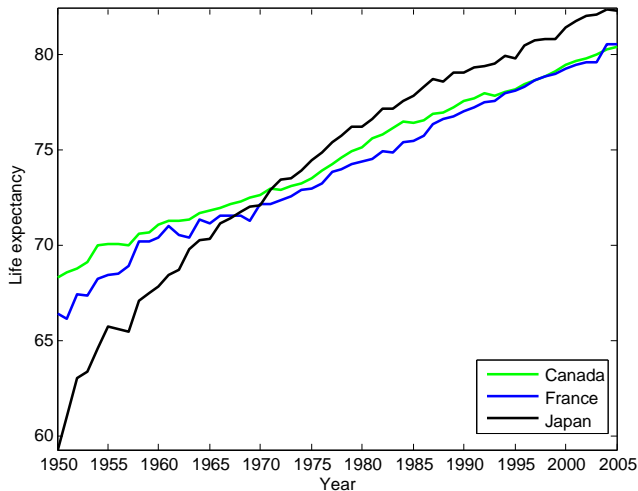
$$k(t, 1) = c + k(t-1, 1) + \xi(t, 1),$$

$$k(t, 2) = \alpha + \beta k(t, 1) + u(t).$$

- Demographic meanings:
  - 1 Longevity improvements are governed by only one driving force.
  - 2 There is an equilibrium relation between mortality rates of the two population.
  - 3 There are random departures from the equilibrium relation.

# The Augmented Common Factor Model

A global convergence in mortality levels:



# The Augmented Common Factor Model

- Life expectancy forecasts based on independent Lee-Carter models are diverging.
- The joint-k model does not solve the problem completely.
- Li and Lee (2005) proved that the necessary conditions for non-diverging forecasts:
  - 1  $b(x, 1) = b(x, 2)$  for all  $x$ ;
  - 2 the drift terms of the ARIMA(0,1,0) processes for  $k(t, 1)$  and  $k(t, 2)$  are identical.

# The Augmented Common Factor Model

- The common factor model:

$$\ln(m(x, t, i)) = a(x, i) + B(x)K(t) + \epsilon(x, t), \quad i = 1, 2.$$

- $K(t)$  is modeled by a random walk with drift,

$$K(t) = c + K(t) + \xi(t).$$

- Problem: it will predict zero basis risk.

# The Augmented Common Factor Model

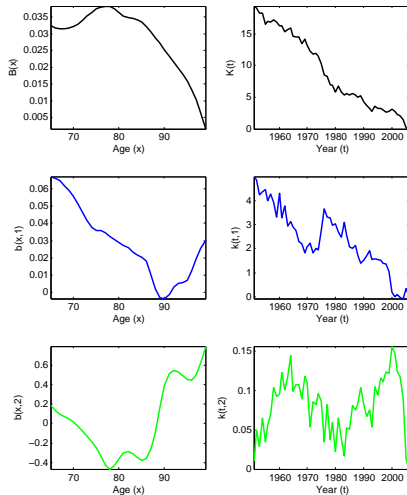
- An improvement – the augmented common factor model:

$$\ln(m(x, t, i)) = a(x, i) + B(x)K(t) + b(x, t)k(t, i) + \epsilon(x, t), \quad i = 1, 2.$$

- $k(t, i)$  is modeled by an AR(1) process:

$$k(t, i) = \phi_0(i) + \phi_1(i)k(t-1, i) + \zeta(t).$$

# The Augmented Common Factor Model



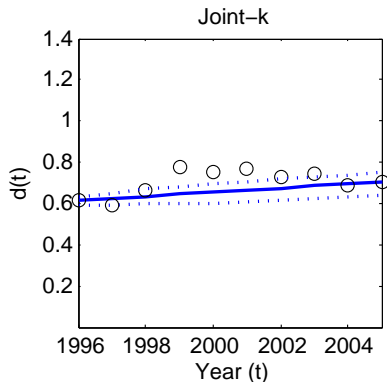
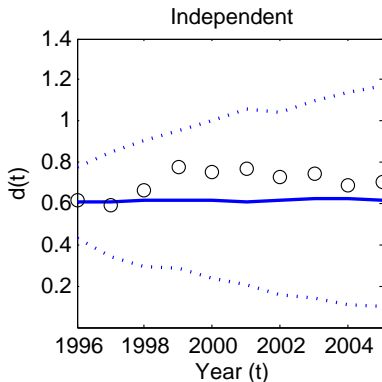
Estimates of parameters in the augmented common factor model.

# Goodness-of-fit

Model	$ER$	$I$	BIC
Independent	0.9891	-36,109	-37,064
Joint-k	0.9884	-40,079	-40,822
Cointegrated	0.9891	-36,109	-37,064
Common factor	0.9655	-58,084	-58,694
Augmented common factor	<b>0.9931</b>	<b>-26,397</b>	<b>-27,697</b>

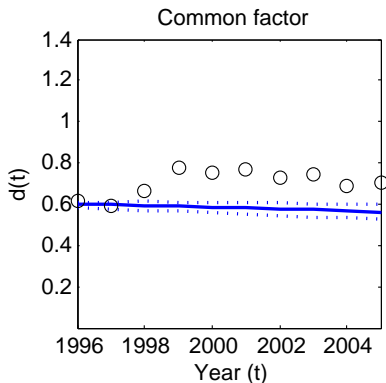
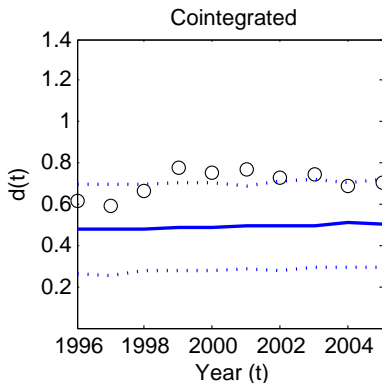
Values of  $ER$ ,  $I$ , and BIC derived from different models.

# Backtesting



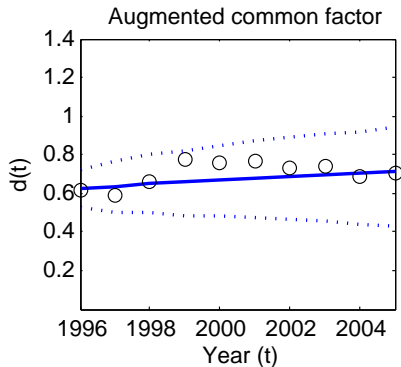
Actual and predicted values of  $d(t) = a(65, t, 1) - a(65, t, 2)$ .

# Backtesting



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# Backtesting

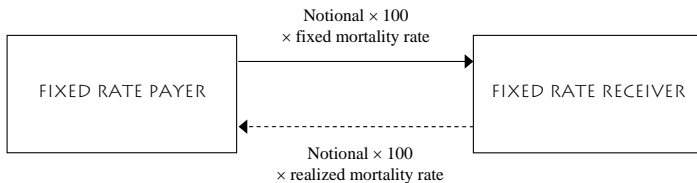


Actual and predicted values of  $d(t) = a(65, t, 1) - a(65, t, 2)$ .

# Hedging Instruments

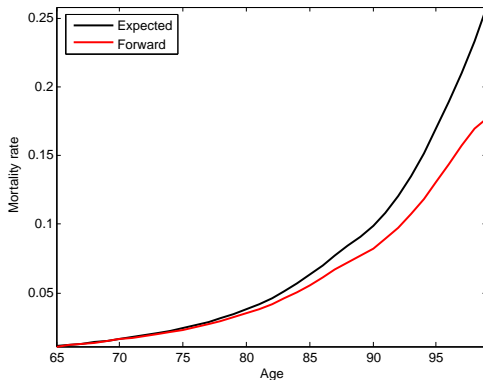
- A combination of q-forwards is used.
- q-forward is a zero-coupon swap that exchanges on the maturity date a fixed amount for a random amount.
- The fixed leg is the 'forward' mortality rate.
- The floating leg is based on the LifeMetrics Index:
  - derived from graduated  $q_x$ ;
  - available for England and Wales, the US, the Netherlands and Germany.

# A q-forward Contract



Settlement of a q-forward contract at maturity.

## 'Forward' Mortality Rates



Expected and forward mortality rates.

# The Hypothetical Plan

- Example: a pension plan that pays \$1 each year to a pensioner who is 65 years old at time 0.
- The pensioner is subject to the same longevity improvements as the female Canadian population.
- No LifeMetrics Index linked to Canadian population mortality.
- US q-forwards are used.
- The hedge is subject to population basis risk.

# Hedging Objectives

- Our goal: to stabilize the P.V. of the cash flows that will be made to the pensioner in 35 years.
- I.e., we want to minimize the *variability* of the *unexpected cashflows*.
- Two measures:
  - 1 **Longevity value-at-risk (VaR)**  
Difference in VaR between the unhedged and hedged positions.
  - 2 **Longevity risk reduction**  
Amount of longevity risk reduction,  $R$ :

$$R = 1 - \frac{\sigma^2(X^*)}{\sigma^2(X)}.$$

## The Basic Idea

- Choose a portfolio of q-forwards that is has a similar sensitivity to changes in the mortality curve.
- How to measure and match sensitivity?
- A single measure is not enough, because:
  - 1 shifts in a mortality curve are mostly non-parallel;
  - 2 a liability has different sensitivities to different parts of the curve.
- However, it is impractical to match sensitivities to death rates at all ages.

# The Hedging Framework

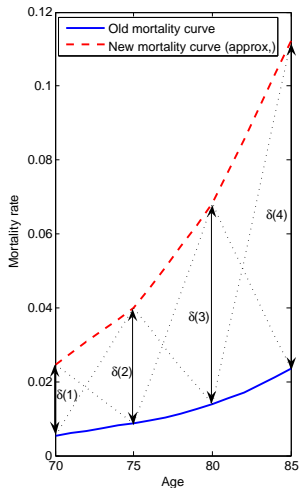
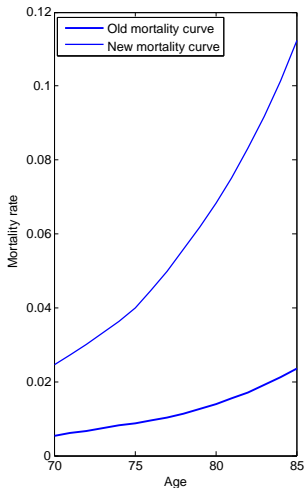
- We use Li and Hardy's (2009) hedging framework.
- It is parallel to Ho's (1992; J. of Fixed Income) method for hedging interest rate risk.

	Ho (1992)	Li and Hardy (2009)
Risk	Interest rate	Longevity
Source	(Spot) yield curve	(Spot) mortality curve
Curve	Not necessarily shift in parallel	
Instrument	Zero-coupon bonds	Standardized q-forwards
Sensitivity	Key rate duration	Key q-duration

# The Hedging Framework

- Key mortality rates:  $q(70, 2011)$ ,  $q(75, 2016)$ ,  $q(80, 2021)$ , and  $q(85, 2026)$ .
- Age dependence means that we can't calculate a key q-duration just by 'shocking' the key rate.
- Key q-durations are calculated as follows:
  - 1 Assume a small change of  $\delta(j)$  in the  $j$ th key rate.
  - 2 Let  $s(x, j, \delta(j))$  be the shift of the mortality curve at age  $x$  associated with  $\delta(j)$ .
  - 3 Calculate  $s(x, j, \delta(j))$  for all  $x$ .
  - 4 Let  $\mathbf{q}$  and  $\tilde{\mathbf{q}}$  be the original and new curves, respectively.
  - 5 We have  $KQD(P(\mathbf{q}), j) = \frac{P(\tilde{\mathbf{q}}) - P(\mathbf{q})}{\delta(j)}$ .

# Illustrating Key q-durations



# The Hedge Portfolio

- Let  $V$  be the p.v. of cash flows that will be made to the pensioner.
- Let  $F_j$  be the p.v. of the payoff from the  $j$ th  $q$ -forward.
- If  $q$ -forwards linked to the population of interest (Population 1) is available, then we need a notional amount of

$$\frac{KQD(V(\mathbf{q}), j)}{KQD(F_j(\mathbf{q}), j)}$$

on the  $j$ th  $q$ -forward,  $j = 1, 2, 3, 4$ .

# The Hedge Portfolio

- If q-forwards linked to Population 2 is used instead, the following adjustment factor is needed:

$$\frac{\partial q(70, 2011, 2)}{\partial q(70, 2011, 1)},$$

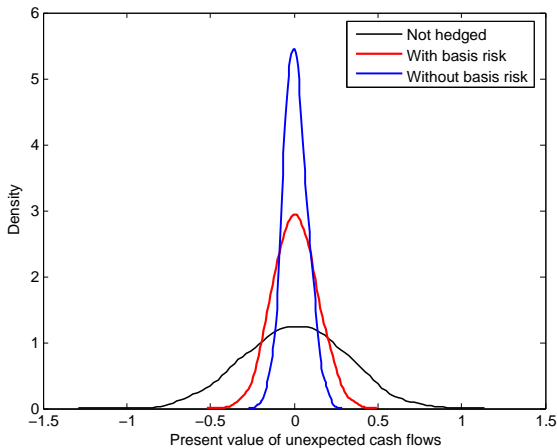
- Under the augmented common factor model,

$$\frac{\partial q(x, t, i)}{\partial q(x, t, j)} = \frac{q(x, t, i)(1 + 0.5m(x, t, j))^2 A(x, t, i)}{q(x, t, j)(1 + 0.5m(x, t, i))^2 A(x, t, j)}$$

where  $A(x, t, i) =$

$$B(x)c + b(x, i)\phi_1(i)^{t-2006}(\phi_0(i) + (\phi_1(i) - 1)k(2006, i)).$$

# Empirical Results



Simulated distributions of unexpected cash flows.

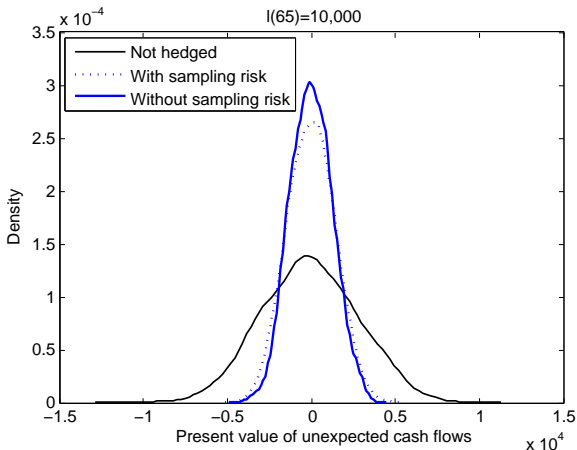
# Empirical Results

- Without a longevity hedge, the VaR is 0.5787.
- With a longevity hedge, the VaR is 0.1544 (without basis risk) and 0.2558 (with basis risk).
- $R = 94.26\%$  in the absence of basis risk.
- $R = 81.61\%$  when there is basis risk.

## Modeling Sampling Risk

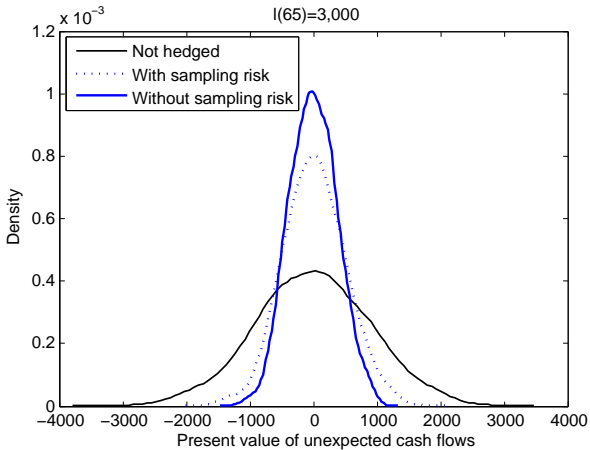
- The previous example assumes no small sample (sampling) risk.
- Sampling risk may affect hedge effectiveness if the plan is not large.
- To measure the impact of sampling risk, we assume  $I(x) \sim \text{Binomial}(I(x-1), 1 - q(x, t, 1))$ .
- The death process is incorporated into the simulations procedure as follows:

## Empirical Results



Simulated distributions of unexpected cash flows.

## Empirical Results



Simulated distributions of unexpected cash flows.

# Empirical Results

	Longevity risk reduction ( $R$ )
$I(65) = +\infty$	81.61%
$I(65) = 10,000$	77.56%
$I(65) = 3,000$	69.57%

## Concluding Remarks

- Four models have been presented.
- The augmented common factor model gives the best goodness-of-fit among the four models.
- A longevity hedge can be reasonably effective, even if population basis risk exists.
- Small sample risk matters, but the hedge is still useful.
- Future research: semi-static/dynamic hedging.

Q&A