An Optimal Strategy of Natural Hedging for a General Portfolio of Insurance Companies

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ABSTRACT

With the improvement of medical and hygienic techniques , life insurers may gain a profit whilst annuity insurers may suffer losses because of longevity risk. In this paper, we investigate the natural hedging strategy and attempt to find an optimal collocation of insurance products to deal with longevity risks for life insurance companies. Different from previous literatures, we use the experienced mortality rates from life insurance companies rather than population mortality rates. This experienced mortality data set includes more than 50,000,000 policies which are collected from the incidence data of the whole Taiwan life insurance companies. On the basis of the experienced mortality rates, we demonstrate that the proposed model can lead to an optimal collocation of insurance products and effectively apply the natural hedging strategy to a more general portfolio for life insurance companies.

Keyword: Longevity risk, Natural hedging strategy, Experienced mortality rates

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1.Introduction

According to the publication-Sigma of Swiss Re, the life expectancy of human around the world will increase 0.2 year per year. Because the mortality rate has improved rapidly for the past decades, longevity risk has become an important topic. In the past two decades, a wide range of mortality models have been proposed and discussed (Lee-Carter, 1992; Brouhns et al., 2002; Renshaw and Haberman, 2003; Koissi et al., 2006; Melnikov and Romaniuk, 2006; Cairn, Blake and Dowd, 2006). Among them, the Lee-Carter (LC) (1992) model is probably the most popular choice, because it is easy to implement and provides acceptable prediction errors.

Constructing delicate mortality model for the use of pricing is one solution to hedge longevity risk for both life insurance and annuity products. However, this solution is often difficult to apply into practice because of market competition. Even though insurance companies have ability to build a delicate mortality model to catch the actual future mortality improvement, they may not be able to price and sell annuity products using the mortality rate derived from this mortality model since it might be too expensive to sell these annuity products with market competition. Another possible solution to hedging longevity risk is to use the mortality derivatives, such as Survival Bonds and Survival Swaps. Blake and Burrows (2001) propose the concept of Survival Bond and insurance companies can hedge longevity risk based on it. Cairn, Blake and Dowd (2006) propose Survival Swaps, which is a contract to exchange cash flows in the future based on the survivor indices. Although mortality derivatives are easy and convenient to use but there are still many obstacles in mortality derivative. The special purpose vehicles must pay more attention on their customers and counterparty, and that means insurance companies have to pay a huge amount of transaction cost on mortality derivatives. Another solution is natural hedging.

Insurance companies can optimize the collocation of its products, annuities and life insurances, to hedge longevity risk. This approach can be done internally in an insurance company. Therefore it is more convenient and practical for insurance company to hedge longevity risk by using this method.

Natural hedging is a relatively new topic in actuarial field, so few papers have studied this issue. Wang, Yang and Pan (2003) investigate the influence of the changes of mortality factors and propose an immunization model to hedge mortality risks. Cox and Lin (2007) indicate that natural hedging utilizes the interaction of life insurance and annuities to a change in mortality to stabilize aggregate cash outflows. And they drew a conclusion that natural hedging is feasible and mortality swaps make it available widely. Wang, Huang, Yang and Tsai (2010) analyze the immunization model mentioned above and use effective duration and convexity to find the optimal product mix for hedging longevity risk. However, their paper uses the same mortality rate for both life insurance and annuity products due to lack of experience mortality data. This is definitely not true in practice.

Different from those previous literatures, we consider the "variance" effect related to both uncertainties of mortality rate and interest rate and the "miss-pricing" effect induced by the mortality improvement in constructing natural hedging portfolio. In addition, we use the experienced mortality rates from life insurance companies rather than population mortality rates. This experienced mortality data set includes more than 50,000,000 policies which are collected from the incidence data of the whole Taiwan life insurance companies. Because we don't have real annuity mortality data, we regard the experience mortality rate of life insurance policies with heavy principal repayment as annuity mortality rate. Both experience mortality rates between with and without principal repayment are correlated - but not perfectly negative correlated.

Therefore, it is not possible to perfectly hedge longevity risk as Wang, Huang, Yang and Tsai (2010) when we consider both life and annuity mortality rates. Besides, we consider the pricing differences in those insurance products between period-mortality basis and cohort-mortality basis. Our objective is to minimize the variation of the change of total portfolio's value and the differences between different pricing bases. On the basis of these experienced mortality rates, the proposed model in this paper appropriately provides an optimal collocation of insurance products and effectively applies the natural hedging strategy to a more general portfolio for life insurance companies.

The reminder of this paper is organized as follows. In Section 2, we review the mortality model and interest rate model, proposing our portfolio model with "variance" effect and "miss-pricing" effect. In Section 3, we calibrate the Lee-Carter model by using the experienced mortality rates from Taiwan Insurance Institute (TII). Section 4 contains the numerical analysis of our model. Section 5 summarizes the paper and gives conclusions and suggestions.

2.Model Setting

2.1. Mortality Model: Lee-Carter Model

Lee and Carter proposed this mortality model in 1992. They found out the fitting and forecasting ability of this model is superior and simple but powerful one-parameter model. Since then, Lee-Carter model has been one of the most popular stochastic mortality models. Their model is as follows:

$$\ln(m_{x,t}) = \alpha_x + \beta_x k_t + \varepsilon_{x,t}, \tag{1}$$

where

$m_{x,t}$: The central death rate for a person aged *x* at time *t*.

α_x: The average age-specific mortality factor.

β_x: The age-specific improving factor.

k_t: The time-varying effect index.

In their study, the time effect index k_t can be estimated by using an ARIMA (0,1,0) process. Note that the continuous-time limit of the ARIMA (0,1,0) process of k_t can be expressed as $dk_t = u_k dt + \sigma_k dZ_k(t)$, where $\left(Z_k(t)\right)_{t=0}^{T}$ is a standard Brownian motion.

There are lots of papers investigating the fitting of Lee-Carter parameters and two most popular methods are SVD (Singular Value Decomposition) and Approximation. In their paper, Lee and Carter use SVD to find the parameters. Wilmoth (1993) propose a modified Approximation method, Weighted Least Squares, to avoid the zero-cell problem. We use Approximation method to estimate the parameters of Lee-Carter model because missing values problem did exist in our data. The computational steps can be written as follows:

1.The fitted values of α_x equal the average of $\ln(m_{x,t})$ at age x over time.

2.Employ the standard normalizing constraints on both β_x and k_t, such that $\sum_x b_x = 1$ and $\sum_t k_t = 0$.

3. k_t is equal to the sum of $\left(\ln(m_{x,t}) - \alpha_x\right)$ over each age.

4.Using the regression model without intercept but with dependent variables k_t and $\left(\ln(m_{x,t}) - \alpha_x\right)$, we can obtain the coefficient β_x.

Following the above steps, we can obtain the estimated parameters α_x, β_x and k_t to forecast the future mortality rates.

2.2. Interest Rate Model: CIR Model

Cox, Ingersoll and Ross (1985) specifies that the instantaneous interest rate follows a square root process, also named the CIR process as follows:

$$dr_t = a(b - r_t)dt + \sigma_r \sqrt{r_t} dZ_r(t), \tag{2}$$

where *a* denotes the speed of mean-reverting adjustment; *b* represents the long term mean of interest rates; σ_r is interest rate volatility; and $\left(Z_r(t)\right)_{t=0}^{T}$ is standard Brownian motion modeling the random market risk factor. In CIR model, the interest rate towards the long run level *b* with speed of adjustment governed by the strictly positive parameter *a*. in this paper, we assume that $2ab \ge \sigma_r^{\ 2}$; therefore, an interest rate of zero is also precluded.

We can transform the continuous-time process into a discrete-time version as follows:

$$r_t = r_{t-1} + a(b - r_{t-1}) + \sigma_r \sqrt{r_{t-1}} \cdot \varepsilon_t, \tag{3}$$

where ε_t is a standard normal random variable. We can use Equation (3) to simulate the future interest rates with the setting origin r_0. The long-term interest rate can be estimated by term structure of interest rate. According to CIR model, we can calculate the price of one unit zero-coupon bond at time t with maturity date *t*+*T* as follows:

$$P(r_t, t, T) = A(t,T) \times e^{-B(t,T) \cdot r_t}, \tag{4}$$

where

$$A(t,T)=\left(\frac{2\gamma e^{[(a+\lambda+\gamma)\cdot T]/2}}{(a+\lambda+\gamma)(e^{\gamma T}-1)+2\gamma}\right)^{2ab/\sigma_r^2}, \tag{5}$$

$$B(t,T)=\left(\frac{2(e^{\gamma T}-1)}{(a+\lambda+\gamma)(e^{\gamma T}-1)+2\gamma}\right), \tag{6}$$

$$\gamma=\sqrt{(a+\lambda)^2+2\sigma_r^2}\ , \tag{7}$$

2.3. General Portfolio Model

In our model, we adopt Lee-Carter mortality model and CIR Interest rate model. The insurance company's portfolio contains zero-coupon bonds, annuities and life insurances with different ages and genders. The factors affecting the total value of this portfolio are mortality rate and interest rate. The model is as follows:

$$V(t)=\sum_{s=A,L}\sum_{g=1,2}\sum_{x=i}^{\omega}N_x^{s,g}V\left(m_x^{s,g}(t),r_t,t\right)+N_BV^B\left(r_t,t\right), \tag{8}$$

where V represents the value of the insurance portfolio; $V^B(r_t,t)$ is the value of one unit of zero-coupon bond with fact value equal to one; $m_x(t)=m_{x+t,t}$; $V(m_x^{A,1}(t),r_t,t)$ ($V(m_x^{A,2}(t),r_t,t)$) denotes the value of one unit of annuity policy issued to the cohort that consists of males (females) aged x at time 0; $V(m_x^{L,1}(t),r_t,t)$ ($V(m_x^{L,1}(t),r_t,t)$) denotes the value of one unit of life insurance policy issued to the males (females) that consists of females aged x at time 0; N_B represents the number of units invested in zero-coupon bonds; and $N_x^{A,g}$ ($N_x^{L,g}$) denotes the number of units allocated in the annuity (life insurance) policies.

From Lee-Carter model, we can get the following information α_x,β_x and k_t by using approximation method and the estimated force of mortality is as follows:

$$\ln(m_{x+t,t}) = \alpha_{x+t} + \beta_{x+t}k_t. \tag{9}$$

Because the continuous-time limit of the ARIMA (0,1,0) process of k_t can be expressed as $dk_t = u_k dt + \sigma_k dZ_k(t)$, differencing Equation (9) yields the continuous -time representation of log mortality rate as follows:

$$d\ln(m_{x+t}) = \beta_{x+t}\left(u_k dt + \sigma_k dZ_k(t)\right), \tag{10}$$

or equivalently,

$$\ln(m^{s,g}_{x+t,t}) = \ln(m^{s,g}_{x+t,0}) + \beta^{s,g}_{x+t}u^{s,g}_k t + \beta^{s,g}_{x+t}\sigma^{s,g}_k Z^{s,g}_k(t)\text{ , s} = L \text{ or } A,\ g = 1\ or\ 2\,. \tag{11}$$

By using the Ito's lemma, we can transform the dynamic of logarithm of mortality into the dynamic of mortality rate as follows:

$$\begin{aligned} d\left(m_{x+t,t}\right) &= d\left(e^{\ln(m_{x+t,t})}\right) = m_{x+t,t}d\left(\ln m_{x+t,t}\right) + \frac{1}{2}m_{x+t,t}d\left(\ln m_{x+t,t}\right)^2 \\ &= m_{x+t,t}\left(\beta_{x+t}u_k dt + \beta_{x+t}\sigma_k dZ_k(t)\right)dt + \frac{1}{2}m_{x+t,t}\left(\beta_{x+t}{}^2\sigma_k{}^2 dt\right) \\ &= \left(m_{x+t,t}\beta_{x+t}u_k + \frac{1}{2}m_{x+t,t}\beta_{x+t}{}^2\sigma_k{}^2\right)dt + \left(m_{x+t,t}\beta_{x+t}\sigma_k\right)dZ_k(t), \end{aligned} \tag{12}$$

or equivalently,

$$d\left(m^{s,g}_x(t)\right) = \left(m^{s,g}_x(t)\beta^{s,g}_{x+t}u^{s,g}_k + \frac{1}{2}m^{s,g}_x(t)\left(\beta^{s,g}_{x+t}\right)^2\left(\sigma^{s,g}_k\right)^2\right)dt + \left(m^{s,g}_x(t)\beta^{s,g}_{x+t}\sigma^{s,g}_k\right)dZ^{s,g}_k(t)$$

$$for\ s = L\ or\ A,\ g = 1\ or\ 2\,, \tag{13}$$

In our model, to capture the covariance structure of mortality rates, we assume that the four mortality risk factors, $Z^{A,2}_k$, $Z^{A,1}_k$, $Z^{L,2}_k$ and $Z^{L,1}_k$ are dependent standard Brownian motions. For the tractability of our analysis, we decompose them into the linear combination of four independent standard Brownian motions $Z_{ki} = \left(Z_{ki}(t)\right)^T_{t=0}$, i=1,2,3,4, by using Chelosky decomposition as follows:

$$\begin{bmatrix} dZ_k^{A,2}(t) \\ dZ_k^{A,1}(t) \\ dZ_k^{L,2}(t) \\ dZ_k^{L,1}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} dZ_{k1}(t) \\ dZ_{k2}(t) \\ dZ_{k3}(t) \\ dZ_{k4}(t) \end{bmatrix}. \tag{14}$$

Utilizing the Ito's lemma, we investigate the change of total insurance portfolio value with respect to the change of mortality rate and interest rate as follows:

$$\begin{aligned} dV(t) &= \sum_{s=L,A}\sum_{g=1,2}\sum_{x=i}^{\omega} N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} dm_x^{s,g}(t) + \frac{\partial V}{\partial r}(dr_t) + \sum_{s=L,A}\sum_{g=1,2}\sum_{x=i}^{\omega} \frac{1}{2} N_x^{s,g} \frac{\partial^2 V_x^{s,g}}{\partial m_x^{s,g\,2}} (dm_x^{s,g}(t))^2 + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(dr_t)^2 \\ &= Q_{0t}\, dt + \sum_{i=1}^{4} Q_{it}\, dZ_{ki}(t) + Q_{5t}\, dZ_r(t), \end{aligned} \tag{15}$$

where $Q_{0t}, Q_{1t}, Q_{2t}, Q_{3t}, Q_{4t}$ and Q_{5t} are defined as follows:

$$\begin{aligned} Q_{0t} = \Bigg\{ &\sum_{s=L,A}\sum_{g=1,2}\sum_{x=i}^{\omega} \Bigg(N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} (m_x^{s,g}(t)\beta_{x+t}^{s,g}\mu_k^{s,g} + \frac{1}{2} m_x^{s,g}(t)\beta_{x+t}^{s,g\,2}\sigma_k^{s,g\,2}) \\ &+ \frac{1}{2} N_x^{s,g} \frac{\partial^2 V_x^{s,g}}{\partial m_x^{s,g\,2}} (m_x^{s,g}(t)\beta_{x+t}^{s,g}\sigma_k^{s,g})^2 \Bigg) + \frac{\partial V}{\partial r} a(b-r) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r{}^2 r \Bigg\}, \end{aligned} \tag{16}$$

$$\begin{aligned} Q_{1t} = \Bigg[&\sum_{x=i}^{\omega} N_x^{A,2} \frac{\partial V_x^{A,2}}{\partial m_x^{A,2}} m_x^{A,2}(t)\beta_{x+t}^{A,2}\sigma_k^{A,2} + a_{21}\sum_{x=i}^{\omega} N_x^{A,1} \frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t)\beta_{x+t}^{A,1}\sigma_k^{A,1} \\ &+ a_{31}\sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t)\beta_{x+t}^{L,2}\sigma_k^{L,2} + a_{41}\sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t)\beta_{x+t}^{L,1}\sigma_k^{L,1} \Bigg], \end{aligned} \tag{17}$$

$$\begin{aligned} Q_{2t} = \Bigg[&a_{22}\sum_{x=i}^{\omega} N_x^{A,1} \frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t)\beta_{x+t}^{A,1}\sigma_k^{A,1} + a_{32}\sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t)\beta_{x+t}^{L,2}\sigma_k^{L,2} \\ &+ a_{42}\sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t)\beta_{x+t}^{L,1}\sigma_k^{L,1} \Bigg], \end{aligned} \tag{18}$$

$$Q_{3t} = \left[a_{33}\sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t)\beta_{x+t}^{L,2}\sigma_k^{L,2} + a_{43}\sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t)\beta_{x+t}^{L,1}\sigma_k^{L,1} \right], \tag{19}$$

$$Q_{4t} = \left[a_{43}\sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t)\beta_{x+t}^{L,1}\sigma_k^{L,1} \right], \tag{20}$$

$$Q_{5t} = \left[\frac{\partial V}{\partial r} \sigma_r \sqrt{r_t} \right]; \tag{21}$$

the $\partial V / \partial r$ and $\partial^2 V / \partial r^2$ are of the form:

$$\frac{\partial V}{\partial r} = \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial r} + N_B \frac{\partial V^B}{\partial r}, \tag{22}$$

$$\frac{\partial^2 V}{\partial r^2} = \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} N_x^{s,g} \frac{\partial^2 V_x^{s,g}}{\partial r^2} + N_B \frac{\partial^2 V^B}{\partial r^2}. \tag{23}$$

Under the assumption that mortality rates and financial risk are independent, the terms $dZ_{ki} dZ_r, i = 1,2,3,4$ are zero in our model. We apply the concepts of effective durations and convexities to estimate the first-order and second-order derivatives in our model as follows:

$$\frac{\partial V}{\partial m} = \frac{V(m^+, r) - V(m^-, r)}{2V(m,r)\Delta m}, \tag{24}$$

$$\frac{\partial V}{\partial r} = \frac{V(m, r^+) - V(m, r^-)}{2V(m,r)\Delta r}, \tag{25}$$

$$\frac{\partial^2 V}{\partial m^2} = \frac{V(m^+, r) + V(m^-, r) - 2V(m,r)}{V(m,r)(\Delta m)^2}, \tag{26}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{V(m, r^+) + V(m, r^-) - 2V(m,r)}{V(m,r)(\Delta r)^2}, \tag{27}$$

where $m^+ = m(1+\upsilon);\ m^- = m(1-\upsilon);\ r^+ = r(1+\upsilon);\ r^- = r(1-\upsilon);\ \upsilon$ represents a strictly positive change rate of mortality rate and interest rate; and Δm and Δr satisfy the following expression:

$$\Delta m = \sum_x \left| m^+ - m \right| \Big/ \sum_x 1, \tag{28}$$

$$\Delta r = \sum_t \left| r^+ - r \right| \Big/ \sum_t 1. \tag{29}$$

By virtue of Equation (15), the variance of the change of total insurance portfolio value is as follows:

$$Var(dV(t)) = \sum_{j=1}^{5} (Q_{jt})^2 \ , \tag{30}$$

which is determined by the parameters of Lee-Carter model and CIR model as well as the first-order and second-order derivatives defined in Equations (24)-(27).

Insurers minimize the variance of portfolio returns with respect to the change in mortality rate and interest rate because their risk profiles are mainly consist of the longevity risk and interest rate risk. Therefore, we incorporate the variance in Equation (30), named as "variance effect", in the objective function for optimally allocating the insurance portfolio with annuity policies and life insurance policies.

Besides, we consider different pricing basis between period-mortality basis and cohort-mortality basis. The mortality rate without the improvement effect is called period mortality rate which lots of insurance companies apply to price insurance policies. The mortality rate with improvement effect, called cohort mortality rate, consists with the tendency of future mortality rate. Consequently, the difference of portfolio value evaluated between these two bases can be regarded as a miss-pricing error. The pricing difference of this portfolio is of the form:

$$D = \left[\sum_{s=A,L} \sum_{g=1,2} \sum_{x=i}^{\omega} N_x^{s,g} \left(V_{period}(m_x^{s,g}(t), r_t, t) - V_{cohort}(m_x^{s,g}(t), r_t, t) \right) \right], \tag{31}$$

where $V_{period}(m_x^{s,g}(t), r_t, t)$ denotes the policy value with period-mortality basis whilst $V_{cohort}(m_x^{s,g}(t), r_t, t)$ denotes the policy value with cohort-mortality basis. As a result, we also incorporate the pricing difference, named as "miss-pricing effect", in the objective function.

The purpose of our paper is to find a feasible policy collocation in the risk profile of insurance companies to minimize the variance effect with a weight (1-θ) and the miss-pricing effect with a weight θ. If the insurance companies put more emphasis on the variance effect, they tend to control the change in the insurance portfolio due to the unexpected shocks in mortality rate and interest rate. If they put more emphasis on the miss-pricing effect, they aim to minimize the miss-pricing error by increasing the weight θ. Our objective function can be expressed as follows:

$$f(N_x^{s,g}, N_B) = \min_{N_x^{s,g}, N_B} \ (1-\theta)\sum_{j=1}^{5} (Q_{jt})^2 + \theta D^2 . \qquad (32)$$

3.Mortality Data

The mortality data were collected via Taiwan Insurance Institute (TII) and include more than 50,000,000 policies of life insurance companies in Taiwan. The original data are categorized by ages, gender and sorts. We use the original data to construct four Lee-Carter mortality tables, including female annuity (fa), male annuity (ma), female life insurance (fl) and male life insurance (ml). The maximal age of the original data is about 85. The parameters of LC model are calibrated by using approximation method and are dipicted in Figure 1.

[Insert Figure **1 here]**

In order to capture longevity risk, we utilize extrapolation to extend maximal age from 85 to 100. For the purpose of forecasting the future mortality rate, in Table 1 we calculate the standard deviation (σ_k) of κ_t^{fa}, κ_t^{ma}, κ_t^{fl} and κ_t^{ml}.

[Insert Table 1 here

We then use Cholesky decomposition method to transform the dependent random variables into a linear combination of independent random variables. First, we compute the correlation matrix (M) of κ_t^{fa}, κ_t^{ma}, κ_t^{fl} and κ_t^{ml} as follows:

$$M = \begin{pmatrix} 1 & 0.774017 & -0.09217 & 0.243753 \\ 0.774017 & 1 & -0.1304 & 0.358255 \\ -0.09217 & -0.1304 & 1 & 0.455803 \\ 0.243753 & 0.358255 & 0.455803 & 1 \end{pmatrix}. \tag{33}$$

Based on the Cholesky decomposition, the lower triangular matrix (R), which satisfies $M = R \cdot R^T$, can be expressed as follows:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7740 & 0.6332 & 0 & 0 \\ -0.0922 & -0.0933 & 0.9914 & 0 \\ 0.2438 & 0.2678 & 0.5076 & 0.7818 \end{pmatrix}. \tag{34}$$

Therefore, based on the lower triangular matrix (R), Equation (14) can be rewritten as follows:

$$\begin{bmatrix} dZ_k^{A,2}(t) \\ dZ_k^{A,1}(t) \\ dZ_k^{L,2}(t) \\ dZ_k^{L,1}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.7740 & 0.6332 & 0 & 0 \\ -0.0922 & -0.0933 & 0.9914 & 0 \\ 0.2438 & 0.2678 & 0.5076 & 0.7818 \end{bmatrix} \begin{bmatrix} dZ_{k1}(t) \\ dZ_{k2}(t) \\ dZ_{k3}(t) \\ dZ_{k4}(t) \end{bmatrix}. \tag{35}$$

4.Numerical Analysis

4.1Scenario 1 : θ = 0 (variance effect)

We first assume that the weight θ is zero, and then we can observe the effect caused by the variance effect only. Whith loss of generity, in the squeal we assume that a=0.1663, b=0.0606 and σ_r=4.733% for the parameters of CIR model. We begin with a simple case, in which there are only two policies in the portfolio: an annuity and a life insurance policy. We can capture the corresponding hedging relation from those simple cases. Given one unit value of annuity due, we can find the optimal value (unit) of life insurance. Assuming that υ (the unexpected change rate of future mortality rate and interest rate) are 5%, 10%, and 15%, Table 2 represents the optimal units of life insurance to hedge one unit value of annuity due for different unexpected change rates.

We see from Table 2 that the unexpected change rate of mortality and interest rate doesn't have obvious impact on the optimal units held in life insurance. The reason is that the effective durations and convexities of each product are almost the same. In Table 3 we take the life insurance policy of female with age 30 (fl30) as an example. The variance increases only slightly with larger unexpected change rate because of the slightly change of duration and convexity. We use 10% unexpected change rate of both mortality and interest rate for the following scenario analyses.

[Insert Table 2 here]

[Insert Table 3 here]

We calculate the corresponding optimal units of different life insurances to hedge one unit of female annuity due. The results are shown in Table 4. From Table 4, we find that we are not able to reduce the total variance by holding female life insurances to hedge annuity policy, but we can make it through male life insurances. Thus, Table 4 shows that the variance of the portfolio of female annuity and male life insurance are smaller than that of the portfolio of female annuity and female life insurance.

[Insert Table 4 here]

We further investigate hold one unit value of male annuity, and the following the optimal units of life insurance with different genders and ages. Table 5 shows the corresponding optimal units of different life insurances to hedge one unit of male annuity due. We find that results are similar to those observed from Table 4. In addition, comparing the right-hand side of Table 4 with that of Table 5, we observe that the optimal units of male life insurance to hedge one unit of male annuity60 (aged 60, male annuity) is larger than those of female annuity60.

[Insert Table 5 here]

We expect to hedge longevity risk of annuity product by holding some units of life insurances because we know that the sign of effective duration of annuity and life insurance are opposite. We can't obtain the hedging effect from holding life insurance because the coefficient of Cholesky decomposition of female life insurance is negative. However, the coefficient of male life insurance is positive, so it is possible to minimize the variance effect by holding some male life insurance policies. The reason that we must hold more male life insurance to hedge male annuity60 than female annuity60 is that the magnitude of male annuity's duration and mortality rate are larger than those of female, as shown in Table 6. Consequently, we must hold more units of male life insurances to hedge 1 unit of male annuity60.

[Insert Table 6 here]

We further want to compare the differences of optimal hedging strategies between holding one unit of annuity due and one unit of deferred annuity. The comparison results are as presented in Tables 7 and 8. From Table 7-1 & 7-2 and Table 8-1 & 8-2, for deferred annuities, we need to hold less life insurance policy to hedge the mortality uncertainty. In addition, we observe that, with the increase of insured's ages, we need less life insurance units to hedge the corresponding annuity. Although the duration of younger life insurances is larger than that of elder, the elder's mortality rate is much higher than the younger's mortality rate. Therefore, it is more possible for the insurance company to suffer an instant claim by the elder insured. So the insurance company needs less units of the elder life insurance to offset the longevity risk of annuities.

[Insert Table 7 here]

[Insert Table 8 here]

4.2Scenario 2: θ=1 (miss-pricing effect)

In this section we ignore the variance effect and discuss the miss-pricing effect. Without consideration of the variance effect in the objective function, we can find a perfect-hedge unit of life insurance product to hedge one unit of annuity, and vice versa. Therefore, the value of objective function is always zero in the optimal situation. In the following analysis, we just show the optimal units of life insurance polices, corresponding to fixedly holding one unit of annuity policy. In Figure 2, we depict the levels of under-pricing for annuity policies and over-pricing for life insurance policies under different ages. We find that, for annuity products, under-pricing problem of male is more serious than that of female. For life insurance products, under-pricing problem of female is more serious than that of male. In addition, we observe that the magnitude of miss-pricing problem decreases as the issued age increases.

[Insert Figure 2 here]

According to Table 9 and Table 10, unlike the result in Scenario 1, we are able to hedge longevity risk of annuity products through female life insurance products in Scenario 2. However, with the issued age increases, the holding tendency goes up. This outcome is totally different from the results in Scenario 1. This implies that insurance companies need to take very different hedging strategies to reduce variance effect and miss-pricing effect respectively.

[Insert Table 9 here]

[Insert Table 10 here]

4.3Scenario 3: 0 < θ <1

In practice, insurance companies should take account of both variance and miss-pricing effects simultaneously to hedge longevity risk. This means the value of θ should between 0 and 1 but not equal to 0 or 1. In this section, we discuss optimal hedging strategies by varying the level of θ. We first consider holding one unit of annuity due and find the optimal units of life insurance. The results of optimal units of female life insurance are expressed in Table 11.

[Insert Table 11 here]

From Table 11, we observe that the optimal units of female life insurance decrease as the weight θ decreases. That is, when we put less emphasis on miss-pricing effect, we tend to minimize the variation of the change of total portfolio value. As a result, the less unit of life insurance we hold, the smaller the objective function is. According to Table 11, the patterns of optimal units of female life insurance to hedge one unit of annuity60 for both male and female under different value of θ are similar, but the magnitude of ma60 is larger than that of female annuity60, which is consistent with as the results in Scenarios 1 & 2 mentioned above. In addition, we find that the female life insurances are unable to reduce the variance of total portfolio value in Scenario 1, which explains the reason why we need less units of life insurance as the weight θ decreases.

In Table 11, insurance companies shall hold smallest optimal units as θ = 0 and hold the largest optimal units as θ =1. When θ is equal to 0.1, 0.01 and 0.005 respectively, the patterns of optimal units are similar to the case with θ =1. In the case of θ = 0.001, we find the pattern of optimal units is similar to the case of θ =1 in younger age, but in older ages the pattern of optimal units will decrease like the case of θ = 0. In Scenario 1, we observe that the variance effect is stronger in older age because the variance effect is positively related to the mortality rate and effective duration. So we will observe the interaction of both variance and difference effect in the case of θ = 0.001 in the following analysis.

In Table 12, we investigate optimal units of male life insurance to hedge one unit of annuity60 for both male and female under different value of θ, and the results are shown as follows. Compared the results in Table 12 with those in Table 11, the optimal units of male life insurance do not always decrease as the weight θ decreases. In Scenario 1, we can utilize male life insurances to reduce total variance term, however, we also take miss-pricing effect into consideration at the same time. Therefore, the outcomes in Table 11 & 12 are the interaction between these two effects.

[Insert Table 12 here]

According to the results of Table 11 and 12, we choose the weight θ=0.001 as an example because there exists an interaction between two effects in average. The results of optimal units of male and female life insurance to hedge one unit of annuity for different ages and genders are expressed in Figure 3. Figure 3 shows that the patterns of optimal units of male and female life insurance to hedge one unit of annuity are similar for different ages and genders. We see from Figure 3 that the trend of optimal units among different ages and genders are similar. The main differences among them are the magnitude of optimal units. In general, optimal units of male life insurance to hedge one unit of annuity is slightly larger than those of female life insurance. Optimal units of male or female life insurance to hedge a younger age of annuity are larger than those to hedge an older age of annuity. In addition, optimal units of male or female life insurance to hedge one unit of male annuity are larger than those to hedge a female annuity.

[Insert Figure 3 here]

5.Conclusion and Suggestion

In this paper, we propose a natural hedging model, which can take account of two important effects of longevity risk at the same time. The first one is the variance of the change of total portfolio value, and the second one is the miss-pricing effect. We can hedge variations of the future mortality rate by the first effect, and hedge the present miss-pricing by the second one.

Previous researches on natural hedging only take account of variance effect, but we incorporate both variance and miss-pricing effects into our model in this paper. In practice, lots of insurance companies indeed have a miss-pricing problem due to mortality improvement over time. Therefore, we contribute to the existing literature by showing that it is not suitable to ignore miss-pricing effect to hedge longevity risk. Unlike the previous literatures, we use the experienced mortality rates from life insurance companies instead of using population mortality rates. These differences make our model more general and easier to apply into practice.

Using the experienced mortality rates, we separate the mortality rate from gender and the type of insurance products, life insurance policies and annuity policies. We use the correlations between these four types of mortality rates to hedge the variations of the future mortality rate. Furthermore, we use Lee-Carter model to forecast the future mortality rate and calculate the level of miss-pricing in practice. Then, insurance companies can decide the relative significance of the variance and miss-pricing effects by a weight θ. So we can effectively apply the natural hedging strategy to a more general portfolio of life insurance companies. When we consider the variance effect only, the optimal units of life insurance is affected by the effective duration and mortality rate mostly. As a result, the optimal units of life insurance decrease as the age of life insurer increases. However, when we consider the miss-pricing effect only, the optimal units are totally decided by the period-cohort difference of each product. The optimal collocation strategy decided by considering the variance effect comes to an opposite conclusion by considering the variance effect case. In this paper, we obtain optimal collocation solutions for both effects to hedge longevity risk. Furthermore, the model proposed in this paper is easy to extend to a general portfolio with realistic combinations of annuity and life insurance policies.

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TABLE 1

Standard deviation of k_t in four different mortality groups

	κ_t^{fa}	κ_t^{ma}	κ_t^{fl}	κ_t^{ml}
Std.	9.34057	10.75928	9.89995	8.26466

TABLE 2

Optimal units of life insurances with different υ ($\theta = 0$)

1 unit fa60			
		fl30	ml30
υ=5%	N	4.51×10^{-5}	0.1353
	$f(N)$	1.65×10^{-7}	1.56×10^{-7}
υ=10%	N	4.51×10^{-5}	0.1353
	$f(N)$	1.66×10^{-7}	1.56×10^{-7}
υ=15%	N	4.51×10^{-5}	0.1353
	$f(N)$	1.66×10^{-7}	1.56×10^{-7}

1 unit ma60			
		fl30	ml30
υ=5%	N	4.51×10^{-5}	0.5411
	$f(N)$	1.23×10^{-6}	1.07×10^{-6}
υ=10%	N	4.51×10^{-5}	0.5411
	$f(N)$	1.23×10^{-6}	1.07×10^{-6}
υ=15%	N	4.51×10^{-5}	0.5412
	$f(N)$	1.23×10^{-6}	1.07×10^{-6}

TABLE 3

Effective duration (dur) and convexity (conv) with different υ (5%, 10%, 15%)

fl30	υ=5%	υ=10%	υ=15%
dur(m)	11.4104	11.4160	11.4255
dur(r)	-49.5854	-50.2119	-51.2687
conv(mm)	-133.783	-133.897	-134.087
conv(rr)	2,701.095	2,719.847	2,751.366
conv(mr)	-348.594	-349.216	-350.201

TABLE 4

Optimal units of life insurances, given 1 unit of female annuity (θ = 0)

1 unit fa60		
	N	*f(N)*
fl30	4.51×10^{-5}	1.66×10^{-7}
fl40	4.51×10^{-5}	1.66×10^{-7}
fl50	4.51×10^{-5}	1.66×10^{-7}
fl60	4.51×10^{-5}	1.66×10^{-7}

1 unit fa60		
	N	*f(N)*
ml30	0.1226	1.56×10^{-7}
ml40	0.0863	1.62×10^{-7}
ml50	0.0407	1.62×10^{-7}
ml60	0.0355	1.62×10^{-7}

TABLE 5

Optimal units of life insurances, given 1 unit of male annuity (θ = 0)

1 unit ma60		
	N	f(N)
fl30	4.51×10^{-5}	4.88×10^{-7}
fl40	4.51×10^{-5}	1.22×10^{-6}
fl50	4.51×10^{-5}	1.22×10^{-6}
fl60	4.51×10^{-5}	1.22×10^{-6}

1 unit ma60		
	N	f(N)
ml30	0.5389	1.06×10^{-6}
ml40	0.3887	1.06×10^{-6}
ml50	0.1457	1.06×10^{-6}
ml60	0.1376	1.06×10^{-6}

TABLE 6

Effective duration and mortality rate of each product (insured)

effective duration				
	30	40	50	60
fa	-3.2065	-2.9768	-2.4200	-1.8099
ma	-4.1073	-3.7589	-2.9648	-2.1000
fl	11.4160	10.4906	8.3051	6.6572
ml	13.3312	10.7881	8.0193	5.8648

mortality rate				
	30	40	50	60
fa	0.000116	0.000168	0.000424	0.001306
ma	0.000121	0.000436	0.001125	0.003044
fl	0.000168	0.000382	0.001155	0.002655
ml	0.000261	0.001	0.002606	0.006274

TABLE 7-1 female annuity due **TABLE 7-2 female deferred annuity**

Comparison of deferred effect of female annuity ($\theta = 0$)

1 unit fa60		
	N	f(N)
ml30	0.1226	1.56×10^{-7}
ml40	0.0863	1.62×10^{-7}
ml50	0.0407	1.62×10^{-7}
ml60	0.0355	1.62×10^{-7}

1 unit fa30		
	N	f(N)
ml30	0.0140	2.04×10^{-9}
ml40	0.0097	2.04×10^{-9}
ml50	0.0046	2.04×10^{-9}
ml60	0.0040	2.04×10^{-9}

TABLE 8-1 male annuity due **TABLE 8-2 male deferred annuity**

Comparison of deferred effect of male annuity ($\theta = 0$)

1 unit ma60		
	N	f(N)
ml30	0.4903	1.07×10^{-6}
ml40	0.338	1.07×10^{-6}
ml50	0.1595	1.07×10^{-6}
ml60	0.1393	1.07×10^{-6}

1 unit ma30		
	N	f(N)
ml30	0.0438	8.54×10^{-9}
ml40	0.0302	8.54×10^{-9}
ml50	0.0142	8.54×10^{-9}
ml60	0.0124	8.54×10^{-9}

TABLE 9

Optimal units of life insurances, given 1 unit of annuity ($\theta = 1$)

1 unit fa60			
fl	N	ml	N
fl30	0.0985	ml30	0.1257
fl40	0.1519	ml40	0.1876
fl50	0.1987	ml50	0.2735
fl60	0.3293	ml60	0.4236

1 unit ma60			
fl	N	ml	N
fl30	0.173	ml30	0.2207
fl40	0.2668	ml40	0.3294
fl50	0.3488	ml50	0.4802
fl60	0.5782	ml60	0.7438

TABLE 10

Comparison of gender effect of annuity ($\theta = 1$)

1 unit fa30			
fl	N	ml	N
fl30	0.3054	ml30	0.3895
fl40	0.4709	ml40	0.5813

1 unit ma30			
fl	N	ml	N
fl30	0.5635	ml30	0.7188
fl40	0.869	ml40	1.0728

fl50	0.6157	ml50	0.8476	fl50	1.1363	ml50	1.5642
fl60	1.0205	ml60	1.3129	fl60	1.8833	ml60	2.423

TABLE 11-

Optimal units of female life insurance to hedge one unit of annuity60 for both male and female under different value of θ

1 unit fa60					1 unit ma60				
θ	fl30	fl40	fl50	fl60	θ	fl30	fl40	fl50	fl60
0	4.51×10^{-5}	4.51×10^{-5}	4.51×10^{-5}	4.51×10^{-5}	0	4.51×10^{-5}	4.51×10^{-5}	4.51×10^{-5}	4.51×10^{-5}
0.0001	0.0979	0.0628	0.0096	4.51×10^{-5}	0.0001	0.1413	0.0773	4.51×10^{-5}	4.51×10^{-5}
0.001	0.0985	0.1362	0.1058	0.0825	0.001	0.1696	0.2333	0.1721	0.1244
0.005	0.0985	0.1486	0.1703	0.2132	0.005	0.1723	0.2596	0.2948	0.3647
0.01	0.0985	0.1502	0.1835	0.2597	0.01	0.1727	0.2632	0.3200	0.4502
0.1	0.0985	0.1518	0.1972	0.3215	0.1	0.1730	0.2664	0.3460	0.5639
1	0.0985	0.1519	0.1987	0.3293	1	0.1730	0.2668	0.3488	0.5782

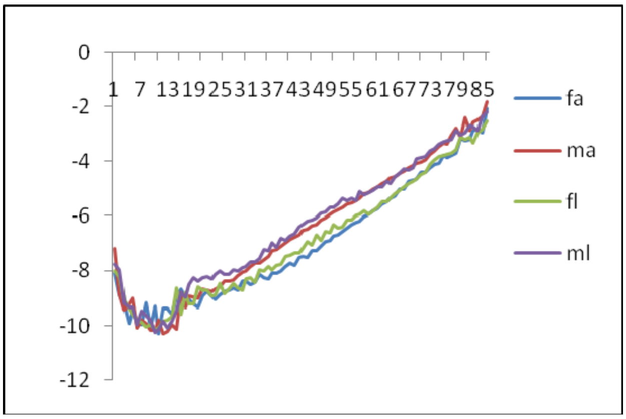
TABLE 12

Optimal units of male life insurance to hedge one unit of annuity60 for both male and female under different value of θ

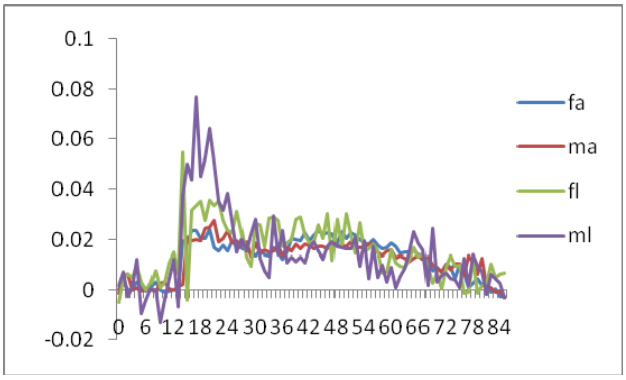
1 unit fa60					1 unit ma60				
θ	ml30	ml40	ml50	ml60	θ	ml30	ml40	ml50	ml60
0	0.1226	0.0863	0.0407	0.0355	0	0.4903	0.338	0.1595	0.1393
0.0001	0.1249	0.1269	0.0560	0.0440	0.0001	0.2857	0.3345	0.1805	0.1525
0.001	0.1256	0.1744	0.1368	0.1064	0.001	0.229	0.3305	0.2919	0.2497
0.005	0.1257	0.1847	0.2221	0.2407	0.005	0.2224	0.3296	0.4094	0.4589
0.01	0.1257	0.1861	0.2447	0.3044	0.01	0.2215	0.3295	0.4406	0.5581
0.1	0.1257	0.1874	0.2706	0.4086	0.1	0.2207	0.3294	0.4762	0.7204
1	0.1257	0.1876	0.2735	0.4236	1	0.2207	0.3294	0.4802	0.7438

FIGURE 1

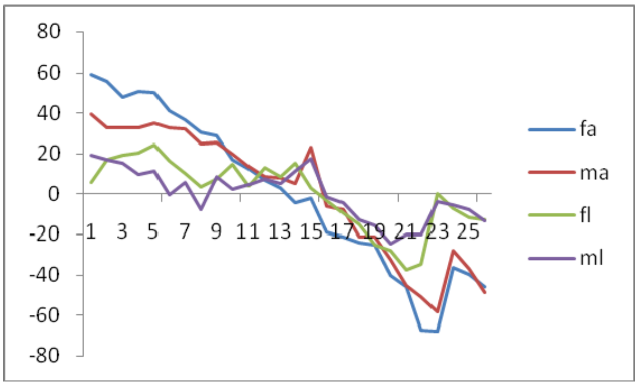
Parameter Estimates of α_x , β_x, k_t in LC model



Panel A. Parameter Estimates of α_x



Panel B. Parameter Estimates of β_x



Panel C. Parameter Estimates of k_t

FIGURE 2

Pricing differences of each product between period and cohort bases

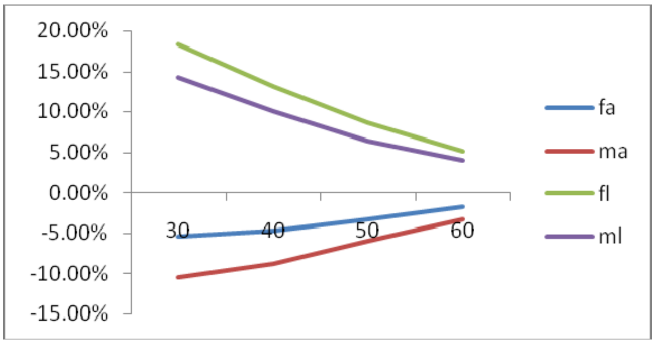


FIGURE 3

Optimal units of male and female life insurance to hedge one unit of annuity for different ages and genders