Pricing of Mortality-linked Contingent Claims: an Equilibrium Approach

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Abstract

This paper introduces an equilibrium approach by assuming that the underlying (motrality rate) has transformed normal distribution to price the Swiss Re mortality bond in a discrete time economy. Our study assumes a general distribution, which is more plausible for valuating mortality-linked securities when the distributions of mortality rate are highly skew. The valuation relationship is still risk-neutral (preference-free) and the mortality bond is priced as the expected value of its terminal payoff, discounted by the riskfree rate. This pricing rule complements current researches on the valuation of mortality-linked securities, which may only exists discrete trading opportunities and insufficient market trading data. Finally, this study gives a closed-form solution for pricing the Swiss Re mortality bond. The mortality bond is a good deal for the investors if the default risk and premium fees can be ignored.

Keywords: longevity risk, mortality-linked securities valuation, transform normal distribution.

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1 Introduction

Longevity and mortality catastrophe risk has created new challenges for financial intermediations, especially for life insurers, reinsurers, annuity providers, and pension funds. This risk is systematic, long-trend, and widespread all over the world. To mitigate uncertain losses, the insurance industry has begun to securitize mortality-related exposures to capital markets. We name these securities as mortality-linked contingent claims (MLCCs).

The related literature for developing valuation methods on MLCCs has grown rapidly in recent years. Two main valuation approaches are the Wang transform (Wang, 2000, 2002) and arbitrage-free pricing of Cairns, Blake, and Dowd (2006b). Wang's approach provides a distortion operator that transforms the underlying distribution to an equivalent risk-adjusted one, applied to discount the expected cash flows with a risk-free rate. Lin and Cox (2005) and Cox, Lin and Wang (2006) use this approach to successfully price the Swiss Re survivor bond. Denuit, Devolder, and Goderniaux (2007) used a similar treatment on the Lee-Carter mortality process to price Survivor bonds.

The approach of Cairns et al. (2006b) uses the arbitrage-free principle to price the European Investment Bank (EIB) longevity bond in an incomplete market setting. Their arbitrage-free pricing approach states that if the market is arbitrage free, there exist one risk-neutral measure Q to calculate fair prices. They assume that the market price of longevity risk is constant and estimate it from the longevity risk premium, implied by the proposed issue price of the EIB longevity bonds. Milevsky and Promislow (2001), and Cairns et al. (2006a) employe similar assumptions and treatments. Bauer et al. (2010) recently compared and commented on these two different approaches.

The concern in this paper is that the MLCC market may only exist discrete trading opportunity and the replicated payoff of MLCCs can not be formed. Besides, up to now, sufficient transaction data in the MLCC market is not available. These market structure and restrictions inspire us a third approach to price the MLCCs.

Rubinstein (1976) and Brennan (1979) propose the valuation relationship in a equilibrium setting that expected cash flow of contingent claims to be discounted at a risk-free rate and is named as risk neutral valuation relationship (RNVR). They also derived the option pricing formulation by assuming that the representative agent has a CRRA/CARA preference; aggregate wealth and underlyings have a joint lognormal/joint normal distribution. Their pricinig formula is the same to the Black-Scholes model (1973) and also preference-free. Doherty and Garven (1986) followed their insights to derive the fair rate of return for the property-liability insurance company. Their results work, depending on the triplet assumptions (preference, wealth, and underlying distributions) and do not require transaction data. Therefore, following their framework, we provide another valuation approach to price MLCCs that is different to Wang's transform and the no-arbitrage method.

One problem in applying the RNVR approach to price MLCCs is that the underlying assets may not be lognormal or normal distributions when longevity risk and catastrophe risk accompany the mortality process. Lin and Cox (2008) showed that when mortality processes with jumps, the terminal loss distribution tends toward positive skew (cf. Lin and Cox, 2008). This article introduces a transformed normal distribution to accommodate high-order moments of mortality risk, especially catastrophe risk, to the MLCCs. The transformed normal distribution includes lognormal and skew lognormal, and the S_U system can have negative, zero, or positive skewness and is more leptokurtic than the lognormal distribution (Johnson (1949) and Johnson, Kotz, and Balakrishnan (1994)). The transformed normal distribution still keeps the RNVR property and the Black-Scholes-type option pricing formula, shown by Camara (2003). With this generalization, we can price MLCCs with a more plausible distributional assumption and devote attention to the high-order moments of mortality risk. We give an example on pricing the Swiss Re mortality bond in Section 4.

Under the triplet assumptions, we use the RNVR for arbitrary MLCCs and their prices are the expected end-of-period payoffs discounted at risk-free rate, taken with respect to the risk-neutral transformed normal density. These results are valuable for current MLCC studies in at least three aspects. First, we find the convenient pricing rule in a equilibrium model, which may only exists discrete trading opportunity and insufficient market transaction data. Second, we could include the default risk as limited liability in option pricing model, which may be crucial in the securitization market. Third, we keep the high-order moments of the original (physical) distribution in the valuation formula, information important for MLCCs valuation. The meaning of transform in our approach is different to Wang's transform, which distorts the original distribution, but we do not. We assume the distribution in a general form that can be transformed into a normal distribution without changing the underlying distribution.

The remainder of this article is organized as follows: Section 2 sets the valuation methods in a discrete time economy, including decomposition of the Swiss Re mortality bond, the valuation methods, and how the RNVR is obtained. Section 3 introduces the risk-neutral valuation approach to price the Swiss Re mortality bond with specified underlying distributions. We derived the colsed-form formualtion under transformed normal distribution. By means of the simulated mortality distribution, Section 4 determine the price of price the Swiss Re mortality bond. An simple approach are provided to esitmate the transformed normal distribution and their parameters. Section 5 offers some conclusions and implications.

2 Mortality-linked Contingent Claim Valuation

2.1 The decomposition of Swiss Re Mortality Bond

The Swiss Re insurance Company issued a three-year mortality bond in December 2003, through a special purpose vehicle (SPV), Vita Capital. The total amount was \$400 million and bondholders receive coupons quarterly at a rate of three-month U.S. dollar LIBOR plus 135 basis points. The principle is not fully protected and is dependent on the mortality index weighted by five countries' mortality experiences.[1] If the mortality index q_t exceeds the 130% of the 2002 level q_0, the principal loss will increase 5% for every 1% raise in the index. If q_t exceeds 150% of q_0, the principal is exhausted. The percentage loss of principal in year t, can be described as:

$$L_t = \begin{cases} 0\% & \text{if } q_t < 1.3q_0 \\ (q_t - 1.3q_0)/0.2q_0 & \text{if } 1.3q_0 \le q_t \le 1.5q_0 \\ 100\% & \text{if } q_t > 1.5q_0. \end{cases} \tag{1}$$

The principal paid back to the bondholders at maturity is:

$$B_T = Max\left(100\% - \textstyle\sum_t L_t,\ 0\right) \tag{2}$$

where t = 1, 2, 3 for year 2003, 2004 and 2005, respectively.

Figure 1 draws the payoff pattern of the principal on t =1. Figure 1 (a) shows that the principal loss is in form of a bear spread depending on mortality level. As we known, the bear spread can be replicated by buying a call option with one strike price and selling another call option with different strike price. Figure 1 (b) shows the loss pattern in solid line, created by the other two calls by buying a call option with a strike price $K_1 = 1.3q_0$ and selling a call option at another strike price $K_2 = 1.5q_0$ at the same time. The aggregate payoff of two calls in Figure 1 (b) is identical to the payoff in Figure 1 (a).

[1] The weights of five countries are the United States (70%), the United Kingdom (15%), France (7.5%), Italy (5%), and Switzerland (2.5%). The index also has weights on males (65%) and females (35%) for each country.

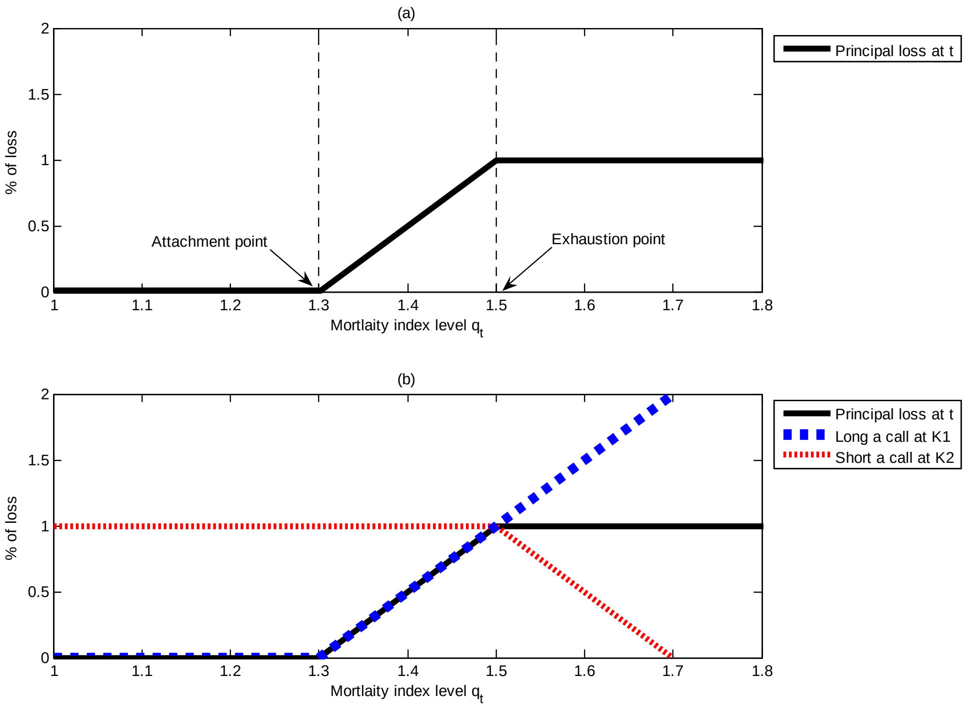

Figure 1 Terminal loss of Swess Re mortality bond and call options. (Revised from Blake, Cairns and Dowd, 2006a)

Thus the principle loss at $t = 1$ of the mortality bond can be formulated by two call options:

$$L_1 = Max\left(\frac{q_1 - K_1}{K_2 - K_1}, 0\right) - Max\left(\frac{q_1 - K_2}{K_2 - K_1}, 0\right) \tag{3}$$
$$= \frac{Max\,(q_1 - K_1, 0) - Max\,(q_1 - K_2, 0)}{K_2 - K_1}.$$

If we find the value of these calls, we find the value of L_1. Similarly, the principle loss at $t = 2$, L_2 has the same exercise price but with a newer mortality level. Replacing q_1 by q_2 in equation (3), we obtain L_2. Again, L_3 can also be obtained by replacing q_1 with q_3 in equation (3). The cumulated loss percentage will be $\sum_t L_t = L_1 + L_2 + L_3$. We can find the terminal principal value at time 0 with some proper discounting:

$$B_0 = 400 \text{ million} \times PV\left[Max\left(100\% - \textstyle\sum_t L_t,\ 0\right)\right] \tag{4}$$

where PV is the present value operator. Section 4 uses equation (4) to find out the price of cumulated loss and the residual value of principle. Some assumptions simplify the multi-period model into an single-period model. Lin, Cox, and Wang (2006), Lin and

Cox (2008), and Chen and Cox (2009), propose an approximation method on $\sum_t L_t$ as follows:

$$\sum_t L_t = \frac{Max\left(q_{\max} - K_1, 0\right) - Max\left(q_{\max} - K_2, 0\right)}{K_2 - K_1} \tag{5}$$

where $q_{\max} = Max(q_1, q_2, q_3)$. Since the probability of two mortality catastrophes at sequence-years are rare, they choose the maximum value of q_t for a representative mortality level. This simplification gives a snapshot of a muti-period valuation as single-period one. For more details, please see Chen and Cox (2009). In Section 4, we uses both method (with and without approximation) to price the principle of the Swiss Re mortality bond.

2.2 General Equilibrium Pricing Model

This subsection reviews the equilibrium pricing on the contingent claims with uncertain payoffs. Let $E^P[\cdot]$ be the expected value operator under the actual probability measure, and U_0 and U_1 are the utility function of a representative investor over the consumptions. The current consumption is C_0; initial wealth and the end-of-period wealth are W_0 and W_1. $P_{j0}(q)$ is the current price of security j written on the underlying q. $P_{j1}(q)$ is the payoff of the security j written on the underlying q at the end-of-period. The demand for the securities is y_j. Assume that the representative agent is non-satiated and risk-averse. By maximizing his expected utility:

$$\underset{C_o,\, y_j}{Max} \;\; U_0(C_0) + E^P \left\{ U \left[\left(W_0 - C_0\right) e^r + \textstyle\sum_{j=1}^{n} y_j (P_{j1}(q) - P_{j0} e^r) \right] \right\},$$

and following the equilibrium condition, we have

$$P_{j0}(q) = e^{-rT} \frac{E^P\left[U'(W_1) P_{j1}(q)\right]}{E^P\left[U'(W_1)\right]} = e^{-rT} E^P\left[\phi(q) P_{j1}(q)\right] \tag{6}$$

where $W_1 = \left(W_0 - C_0\right) e^r + \sum_{j=1}^{n} y_j (P_{j1}(q) - P_{j0} e^r)$ and $\phi(q)$ is pricing kernel defined as:

$$\phi(q) = \frac{U'(W_1 \,|q)}{E^P\left[U'(W_1)\right]}, \tag{7}$$

Equation (6) states the price of any security may be expressed as the expected value product by its relative conditional marginal utility of wealth and discounted at the riskfree interest rate. Specifically, throughout the rest of this article, we focus on one underlying (i.e., mortality rate) and one contingent claim written on it. Then, the subscripts j in equation (6) can be suppressed and rewritten as:

$$P_0 = e^{-rT} E^P\left[\phi(q) P_1(q)\right] \tag{8}$$

In equation (8), the price relationship is still preference-dependence.

2.3 Risk Neutral Valuation Relationships

To analyze the expected value of equation (8), we need to specify the distributions of terminal wealth, underlying asset, and represent agent's utility. Camara (2003) propose the transformed-normal distribution:

DEFINITION *The transformed normal distribution are defined by the transformation of random variable q such that:*

$$f(\frac{q-\alpha}{\beta}) = x \backsim N(\mu, \sigma^2) \tag{9}$$

where α, β, μ and σ are parameters ($\beta, \sigma > 0$) and f is a strictly monotonic differential function. $N(\mu, \sigma^2)$ is a normal distribution with mean μ and variance σ^2.

This definition is identical to the four-parameter transformed normal distribution of Johnson (1949) and Johnson, et al. (1994), but slightly different to the three-parameter transformed normal distribution of Camara (2003). We adapt Johnson's definition because we find it have a better fitting result than Camara's definition in our mortality data. Transformed normal distribution are much more general than normal distribution. There are some well-known distribution can be included in transformed normal distribution. For example, if $\alpha = 0$, $\beta = 1$ and f is log function, then q is a lognormal distribution. If $q > \alpha > 0$, $\beta = 1$ and f is log function then q is a skew lognormal distribution.

Assume the terminal wealth W_1 and underlying q have a joint distribution:

$$(f(W_1),\ f_1(q)) \sim \mathbf{N}(\mu_w, \mu, \sigma_w, \sigma, \rho) \tag{10}$$

where f and f_1 are a strictly monotonic differentiable function as defined in equation (9), $\mathbf{N}$ denotes the bivariate normal distribution with means μ_w and μ; standard deviations σ_w and σ; and correlation coefficient ρ. The subscripts $_w$ denotes the parameters for the wealth. The marginal utility of the represent agent is in the form:

$$U'(W_1) = \exp^{\beta f(W_1)} \tag{11}$$

where β is constant and f is the same one as in equation (10). Following the assumption of (10) and (11), the equilibrium price is:

$$P_0 = e^{-rT} E^Q \left[P_1(q)\right]$$

where $E^Q\left[\cdot\right]$ is the expected value operator under the Q probability measure with respect to the risk-neutral transformed normal density.[2] This risk-neutral density has a shifted location of the underlying density and does not relate to preference parameters. Therefore, the value of the contingent claim is the expected cashflow of $P_1(q)$ discounted by risk-free rate. We now have a RNVR to price MLCCs and discount their future contingent payoff by the risk-free rate.

This RAVR places restrictive assumptions on the underlying distribution, individual wealth, and preference. The no-arbitrage pricing does not need such assumptions. However, when a replicated portfolio is not available and lacking of trading data for MLCCs, equilibrium pricing still provides information. The transformed normal distribution keeps high-order moments in the pricing formula, such as skewness. The next section shows that the high-order moments affect the valuations.

3 Valuation of The Swiss Re Mortality Bond

3.1 Specifications of underlying distribution

In this section, we present some more specific distribution of transformed normal distribution that lead to closed-form solutions as Black-Scholes-type formula for the Swiss Re Mortality Bond. We assume q follow (a) S_U distribution, (b) skewed lognormal distribution and (c) lognormal distribution and derive their corresponding pricing formula.

(a) Assume that aggregate wealth has a lognormal distribution; the mortality rate q follows the S_U system developed by Johnson (1949) and representative agent has a power utility function displaying constant relative risk aversion (CRRA). Thus $f(W_1) = \ln(W_1)$ and $f_1(q) = \sinh^{-1}(\frac{q-\alpha}{\beta}) = \ln(\frac{q-\alpha}{\beta} + \sqrt{1+(\frac{q-\alpha}{\beta})^2}) \backsim N(\mu, \sigma)$ and

$$\left(\ln(W_1), \sinh^{-1}(\frac{q-\alpha}{\beta})\right) \backsim N(\mu_w, \mu, \sigma_w, \sigma, \rho)$$

where β and σ are positive constants. With the terminal payoff $Max(q-K, 0)$, the option price at $t = 0$ is:

$$P_0 = \frac{\beta}{2}e^{-rT+\mu^Q+\frac{1}{2}\sigma^2 T} \cdot \Phi(d_1) - \frac{\beta}{2}e^{-rT+\mu^Q+\frac{1}{2}\sigma^2 T} \cdot \Phi(d_2) + (\alpha - K)e^{-rT} \cdot \Phi(d_3) \quad (12)$$

[2]For the proof, please see the proposition 2 of Camara (2003).

where

$$
\begin{aligned}
\mu^Q &= \sinh^{-1}\left(\tfrac{1}{\beta}e^{-\frac{1}{2}\sigma^2 T}(q_0 e^{rT}-\alpha)\right) \\
d_1 &= \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma\sqrt{T}}+\sigma\sqrt{T} \\
d_2 &= \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma\sqrt{T}}-\sigma\sqrt{T} \\
d_3 &= \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma\sqrt{T}}
\end{aligned} \tag{13}
$$

and $\Phi(\cdot)$ is cumulated standard normal distrubution.
Proof: see Appendix A.

(b) Assume that the aggregate wealth has normal distribution, the mortality rate q_t has a skewed lognormal distribution, and the agent has a negative exponential preferences. Thus $f(W_1) = W_1$ and $f_1(q_t) = \ln(q-\alpha)$ where α is a positive constant. With the terminal payoff $Max(q-K,0)$, the option price is

$$
P_0 = \left(q_t - \alpha e^{-rT}\right)\Phi(d_1) - e^{-r}\left(K-\alpha\right)\Phi(d_2)
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(\frac{\alpha-K}{\alpha-q_t e^{r(T-t)}}) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 + \sigma\sqrt{T-t}
\end{aligned}
$$

We show the proof in Appendix B.

4 Parameter Estimation and Option Pricing

In this section, we use several stochastic mortality processes to generate the terminal distribution of q. These include Lee-Carter with jumps model (Chen and Cox, 2009), and the catastrophe model of Lin and Cox (2008). We obtained our mortality data from the National Center for Health Statistics (NCHS)[3]. We use this data to generate the mortality distribution from 2003 to 2005, regarding these as discrete distribution at each end-of-year. We estimate the parameters of the distribution according to the transformed normal distribution. The four-parameters transformed normal distribtuion are difficult to estimate in the usual manner. We proposed a quantile-estimation method adapted from Slifker and Shapiro (1980) to facilitate the implementation.

[3] http://www.cdc.gov/nchs/nvss/mortality/

4.1 Parameter Estimation for the S_U distribution

The Slifker and Shapiro (1980) quantile-based estimation method provides two advantages. One is an increase in the accuracy when the observations of data is large enough. The other advantage is that the estimators are explicit formulations, which is easy to implement. Below, we descript the steps that the parameters were estimated.

Consider any data you have on hand: Choose any value $z > 0$ from a standard normal random variable (for example, choose $z = 1$). Then the four points $\pm z$ and $\pm 3z$ determine the corresponding value of the raw data. They are q_{-3z}, q_{-z}, q_z and q_{3z}. Let

$$m = q_{3z} - q_z$$
$$n = q_{-z} - q_{-3z}$$
$$p = q_z - q_{-z}.$$

If the data passes the criteria of Su distribution, i.e. $mn/p^2 > 1$, then the estimates for the parameters are:

$$\alpha = \frac{x_z + x_{-z}}{2} + \frac{n-m}{2(\frac{m}{p} + \frac{n}{p} - 2)};$$

$$\beta = \frac{2p\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}{\left(\frac{m}{p} + \frac{n}{p} - 2\right)\left(\frac{m}{p} + \frac{n}{p} + 2\right)^{1/2}}; \quad (\beta > 0)$$

$$\mu = \sinh^{-1}\left[\frac{\frac{m}{p} - \frac{n}{p}}{2\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}\right];$$

$$\sigma = \frac{\cosh^{-1}\left[\frac{1}{2}\left(\frac{m}{p} + \frac{n}{p}\right)\right]}{2z}. \quad (\sigma > 0)$$

The estimated parameters of mortality data from 2003 to 2005 and $q_{\max}$ are shown in the Table 1.

Table 1 The parameters of mortality data assuming following Su distribution

		q_{2003}	q_{2004}	q_{2005}	$q_{\max}$
Basic statistics	mean	0.0079608	0.0072285	0.0065657	0.0079698
	standard deviation	0.0002845	0.0004107	0.00051437	0.0002957
	skewness	1.4226	0.52041	0.36981	1.5167
	kurtosis	10.658	4.6036	3.5588	10.473
Transformed parameters	α	0.0076907	0.0067493	0.0055682	0.0076918
	β	0.0002611	0.0007357	0.0016061	0.0002638
	μ	0.73272	0.55769	0.56733	0.73965
	σ	0.68841	0.44484	0.26635	0.68807

In table 1, we present both the basic statistics and the transformed parameters. In the basic statistics, the row of means shows a decreasing trend from 0.0079608 to 0.0065657. The decreasing trend in mortality rates is a design of Lin and Cox (2008). Maximum value of the three years, $q_{\max}$ shows the greatest mean value, 0.0079698. The row of standard deviations shows an increasing trend from 0.0002845 to 0.00051437, because their model assumes that mortality rate follows geometric Brownian motion and volatility increases with time. However, compared with other years, $q_{\max}$ has a smaller standard deviation (0.0002957). We regard the approximation value of mortality rates creates a "censor effect", $q_{\max}$ is no longer the largest of the three in standard deviation. Therefore, if standard deviation is crucial to pricing, this approximation method may undervalue the risk. The row of skewness shows a positive skew in the data, with $q_{\max}$ showing the greatest degree of skewness.

In the transformed parameters, α is a location parameter, showing a decreasing trend from 2003 to 2005, with $q_{\max}$ have the greatest location parameter. The deviation parameter, β, does not show a consistent trend form 2003 to 2005. The other two parameters, μ and σ, are the mean and standard deviation of the normal distribution implying that can be transformed again to obtain standard normal distribution.

4.2 Options and mortality bond prices

In this section, we calculate mortality bond price by assuming the underlying transformed normal distribution of the S_U system. Initial mortality q_{2002} is 0.008758, riskfree interest rate =0.0%[4], time to maturity= 1 years (3 years for $q_{\max}$) and exercise prices $K_1 = q_{2002} \times 1.3$ and $K_2 = q_{2002} \times 1.5$. For a better intuition in price, we assumed the principle is 1,000 as opposed to 400 million. Using equation (12) and transformed parameters in Table 1, the prices of call options and the mortality bond are shown in Table 2:

[4] We discuss our result with regard to a premium spread, therefore the assumption of risk-free rate zero provides a clearer presentation without changing our conclusions.

Table 2 The prices of calls and mortality bond under Su distribution

Distribution	Present value of	Without Approximation			Approximation
		t=2003	t=2004	t=2005	
Su	call 1 = Max(q_t-K_1, 0)	0.020277	0.019170	0.0079474	0.020199
	call 2 = Max(q_t-K_2, 0)	0.005293	0.002997	0.0004304	0.005264
	L_t	8.5544	9.2334	4.2915	8.5267
	Bond price			977.92	991.47
	Par spread			74 bps	29 bps

In Table 2, the results were derived from two methods. One method used approximation; and the other did not. In the method without approximation, the call prices showed a decreasing trend from 0.020277 to 0.007947 because the mean of q_t is decreasing in mean form 2003 to 2005 and the call prices also have a decrease trend form 0.020277 to 0.007947. The exercise price, K_1 is larger than K_2; therefore, the value of call 1 was higher than the value of call 2 in each year. Using these call prices and equation (5), we determine principle losses of L_t =8.5544, 9.2334, and 4.2915 dollars for each year. Following the relation $B_0 = max\left(1 - \sum_t L_t,\ 0\right)$, the mortality bond price is 977.92 and par spread is 74 bps.

In the method that used approximation, the mortality bond price is 991.47 and par spread is 29 bps. This spread is smaller than in the method without approximation (74 bps). Comparing our results with other studies, for example, the spread of Lin and Cox (2008) is 39 bps; and the spread of Chen and Cox is 56 bps. In both of these studies, they applied the method with approximation. We believe that the spread was underestimated using the approximation method. However, the Swiss Re offeres a spread of 135 bps, which are still greater than the spread in our without approximaton method. We must emphasize that our study does not include default risk or premium fees. Therefore, it is hard to say whether the Swiss Re bonds (with a spread of 135 bps), is a good deal for investors.

5 Conclusion and Discussion

In this article, we provide an equilibrium pricing approach to value MLCCs and apply it on the Swiss Re mortality bond. An convenient and explicit valuation formulation is obtained. There are several features in this approach. First, relative to the no-arbitrage pricing of Cairns et al. (2006), we make more restricted assumptions (utility, wealth and underlying distribution). The trade-off is that we do not require market transaction data and replicated portfolio to determine the price of MLCCs. Second, relative to the Wang transform (Wang, 2000, 2002), the valuation is preference-free and the payoff could be discounted at risk-free rate, too. However, our approach does not distort the underlying distribution; we assume a more general distribution that can be transformed into normal distribution. Third, to obtain a better estimation in mortality data, we derive a new option pricing formula for the four-parameter transformed normal distribution, different from that of Camara (2003). A quantile-based estimator of S_U is applied, to facilitate the estimation process. Finally, the transformed normal distribution enables the inclusion of high-order moments, when the mortality jump is important to the valuation.

Based on the results in our study, we formulate two conclusions. First, the approximation method increases the mean, but decreases the variance of the mortality rate. If the variance is important to the valuation, this method would undervalue the mortality bond price. Second, whether using approximation methods or not, both spreads are smaller than the level offered by Swiss Re. We do not consider the default risk or loading fees, and maybe undervalue the price of the mortality bond. However, if the friction costs of MLCC market is small or could be omitted, it may be that Swiss Re over-compensated mortality bond investors.

Appendix A

Let x=$\sinh^{-1}\left(\frac{q-\alpha}{\beta}\right)$ ~$N(\mu,\sigma)$ and $\beta > 0$, $\sigma > 0$. From equation (), the pricing relation is

$$P \cdot e^{rt} = E^Q\left[S(q)\right] = E^Q\left[q - K|\, q \geq K\right].$$

Substituting q in terms of x, yields the result

$$\begin{aligned} P \cdot e^{rt} &= E^Q\left[\alpha + \beta \sinh x - K|\, x \geq \sinh^{-1}\left(\tfrac{K-\alpha}{\beta}\right)\right] \\ &= \int_{\sinh^{-1}(K-\alpha)/\beta}^{\infty} \alpha + \beta \sinh x - K \cdot f\left(x; \mu^Q, \sigma\sqrt{T}\right) dx \\ &= \tfrac{\beta}{2}\int_{\sinh^{-1}(\frac{K-\alpha}{\beta})}^{\infty} e^{x} \cdot f\left(x; \mu^Q, \sigma\sqrt{T}\right) dx - \tfrac{\beta}{2}\int_{\sinh^{-1}(\frac{K-\alpha}{\beta})}^{\infty} e^{-x} \cdot f\left(x; \mu^Q, \sigma\sqrt{T}\right) dx \\ &\quad + (\alpha - K)\int_{\sinh^{-1}(\frac{K-\alpha}{\beta})}^{\infty} f\left(x; \mu^Q, \sigma\sqrt{T}\right) dx \end{aligned}$$

Using the definition of lognormal distribution, it follows that

$$\begin{aligned} P \cdot e^{rt} &= \tfrac{\beta}{2} e^{\mu^Q + \frac{1}{2}\sigma^2 T} \cdot \Phi\left(\frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma\sqrt{T}} + \sigma\sqrt{T}\right) \\ &\quad - \tfrac{\beta}{2} e^{\mu^Q + \frac{1}{2}\sigma^2 T} \cdot \Phi\left(\frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right) + (\alpha - K) \cdot \Phi\left(\frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma\sqrt{T}}\right) \end{aligned}$$

Therefore, the option price is

$$P = \frac{\beta}{2}e^{-rT+\mu^Q+\frac{1}{2}\sigma^2T} \cdot N\left(d_1\right) - \frac{\beta}{2}e^{-rT+\mu^Q+\frac{1}{2}\sigma^2T} \cdot N\left(d_2\right) + (\alpha - K)e^{-rT} \cdot N\left(d_3\right)$$

where

$$\mu^Q = \sinh^{-1}\left(\frac{1}{\beta}e^{-\frac{1}{2}\sigma^2T}(q_0e^{rT} - \alpha)\right)$$
$$d_1 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma\sqrt{T}} + \sigma\sqrt{T}$$
$$d_2 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$
$$d_3 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta})+\mu^Q}{\sigma\sqrt{T}}$$

is the same to equation (13).

Appendix B

Reference

1. Bauer, D., M., Börger and J., Russ, 2010, On the pricing of longevity-linked securities, *Insurance: Mathematics and Economics*, 46:139-149.

2. Biffis, E. (2005). Affine Processes for Dynamic Mortality and Actuarial Valuations, *Insurance: Mathematics and Economics*, 37:443-468.

3. Black, Fisher, and Myron S. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81,637-659.

4. Blake, D., Cairns, A.J.G., and Kevin Dowd (2006a). Living with Mortality: Longevity Bonds and Other Mortality-Linked Securities, *British Actuarial Journal*, 12: 153-197.

5. Blake, D., Andrew Cairns, Kevin Dowd, and Richard MacMinn (2006b). Longevity Bonds: Financial Engineering, Valuation, and Hedging. *Journal of Risk and Insurance*, 73, No.4: 647-672.

6. Brennan, Michael J., 1979, The pricing of contingent claims in discrete time models, *Journal of Finance* 34, 53-68.

7. Cairns, A.J.G., D. Blake and K. Dowd (2006a). Pricing Death: Frameworks for the Valuation and Securitization of Mortality Risk, *ASTIN Bulletin*, 36, 79-120.

8. Cairns, A.J.G., D. Blake and K. Dowd (2006b). A Two-factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration, *Journal of Risk and Insurance*, 73: 687-718.

9. Camara, A., (2003), A generalization of the Brennan-Rubinstein approach for the pricing of derivatives, *Journal of Finance,* 58, 805-819.

10. Chen, H., and , S.H. Cox, (2009) Modeling Mortality with Jumps: Applications to Mortality Securitization, *Journal of Risk and Insurance*, 76, No.43: 727-751.

11. Cox, S.H., and Y. Lin (2007). Natural Hedging of Life and Annuity Mortality Risks, *North American Actuarial Journal*, 11, No. 3, 1-15.

12. Cox, S.H., Y. Lin, and S. Wang (2006). Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization. *Journal of Risk and Insurance*, 73, No.4: 719-736.

13. Denuit, M., P. Devolder, and A. Goderniaux (2007). Securitization of Longevity Risk: Pricing Survivor Bonds with Wang Transform in the Lee-Carter Framework. *Journal of Risk and Insurance*, 74, No.1: 87-113.

14. Doherty, N. A., and Garven, J. (1986). Price regulation in property-liability insurance: A contingent-claims approach, *Journal of Finance,* 41, 1031–1050.

15. Dowd, K (2003). Survivor Bonds: A Comment on Blake and Burrows. *Journal of Risk and Insurance*, 70, No.2: 339-348.

16. Dowd, K., D. Blake, A.J.G. Cairns, and P. Dawson (2006). Survivor Swaps. *Journal of Risk and Insurance*, 1: 1-17.

17. Huang, Chi-fu, and Robert H. Litzenberger, 1988, *Foundations for Financial Economics*, (Prentice-Hall, New York).

18. Johnson, Norman L., 1949, Systems of frequency curves generated by methods of translation, *Biometrika* 36,149-176.

19. Johnson, Norman L., Samuel Kotz, and N. Balakrishnan, 1994, *Continuous Univariate Distributions*, Vol. 1, (Wiley, NewYork).

20. Lin, Y., and Cox, S.H. (2005). Securitization of Mortality Risks in Life Annuities, *Journal of Risk and Insurance*, 72:227-252.

21. Lin, Y., and Cox, S.H., (2008). Securitization of catastrophe mortality risks. *Insurance: Mathematics and Economics* 42, 628–637.

22. Milevsky, M.A., Promislow, S.D., 2001. Mortality derivatives and the option to annuitize. *Insurance: Mathematics and Economics* 29 (3), 299–318.

23. Milevsky, M.A., S.D. Promislow, and V.R. Young (2006). Killing the Law of Large Numbers: Mortality Risk Premiums and the Sharpe Ratio, *The Journal of Risk and Insurance*, 73:4 673-686.

24. MorganStanley, 2003. Swiss Re innovative mortality-based security. See the news release for December 8, 2003.

25. Rubinstein, Mark, 1976, The valuation of uncertain income streams and the pricing of options, *Bell Journal of Economics and Management Science* 7,407-425.

26. Rubinstein, Mark, 1983, Displaced diffusion option pricing, *Journal of Finance* 38, 213-217.

27. Sherris, M., (2006). Solvency, Capital Allocation, and Fair Rate of Return in Insurance, *The Journal of Risk and Insurance*, Vol. 73, No. 1, 71-96.

28. Slifker, J.F. and S.S. Shapiro, 1980, The Johnson System: Selection and Parameter Estimation, *Tecgnometrics*, 22, no. 2, 239-246.

29. Wang, Jennifer L., H.C. Huang, Sharon S. Yang, and Jeffrey T. Tsai, (2008). An Optimal Product Mix for Hedging Longevity Risk in Life Insurance Companies: The Immunization Theory Approach *Journal of Risk and Insurance*, Forthcoming.

30. Wang, S.S., 2000. A class of distortion operations for pricing financial and insurance risks. *Journal of Risk and Insurance* 67 (1), 15–36.

31. Wang, S.S., 2002. A universal framework for pricing financial and insurance risks. *ASTIN Bulletin* 32 (2), 213–234.