PRICING LIFE EXPECTANCY

- A FRAMEWORK FOR LONGEVITY OPTIONS -

by

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ABSTRACT

It is widely accepted that hedging smoothens cash flows of a corporation and thus reduces the cost of financial distress. For many underlying risks such as changes in interest rates, credit spreads, commodity prices or FX rates, financial markets provide derivative solutions for transferring risk. However, the risk of changing mortality rates is more difficult to deal with as appropriate risk management tools barely exist. Life insurances and pension funds are the most obvious, but not exclusive entities being exposed to unexpected changes in mortality. As they underwrite obligations with potentially vast payoffs, it is crucial to establish a well functioning market for longevity derivatives to transfer the risks efficiently.

After briefly introducing the fundamentals of the market for longevity risk, literature is reviewed and summarized before a universal applicable framework to price longevity options is derived and tested for its sensitivities.

KEYWORDS

Longevity derivatives, longevity forward, longevity options, longevity risk, mortality rates, mortality risk, stochastic mortality

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CHAPTER 1:

INTRODUCTION

"I don't want to achieve immortality through my work. I want to achieve it through not dying"[1]; where immortality is a dream of many individuals, increasing life expectancy (LE; basic definitions see Appendix A) creates risk for economic agents such as (corporate) pension funds, insurance companies and governments. Besides the risk of underfunded schemes on the asset side (i.e. actual < expected return), they have in common being overall *short longevity* meaning that increasing LE generates additional liabilities. Its accurate estimation is crucial as it determines the future obligation and misjudgments may lead to huge holes in the balance sheets with devastating implications for the retirement income of each individual. Pensions and annuity providers have experienced strong deviations in mortality over the past decades with mortality assumptions underestimating mortality improvements (Loeys et al., 2007). The Pensions Regulator in the UK said that firms have underestimated how long people live, leaving them with a shortfall of at least GBP 75bn (Chapman, 2008); each additional year of life adds approximately 3-4% to the value of UK pension liabilities (The Pensions Regulator, 2007). However, LEs are difficult to forecast as they are

[1] Woody Allen

surrounded by both uncertainty regarding future mortality improvements and 'shocks' (e.g. war, pandemic).

THE CURRENT STATE

With regards to risk management, the major components from an actuarial perspective are interest rate, inflation and longevity risk, with the former two being well manageable, especially through the emergence of derivatives. In contrast, longevity risk management is in its early stages of research and development. As exposure arises from unexpected changes in LE, affecting the fair value, premium rates and risk reserve calculations of insurers and pensions, it is important to have access to appropriate tools that allow for transferring risk efficiently. There are four principal solutions to mitigate longevity risk; **traditional life reinsurance:** Both financial and longevity risks are passed through; however, reinsurers usually prefer to take on small-sized deals due to the lack of appropriate risk management tools. **(Pension) buy-outs** in various forms have emerged over the past years where companies (partially) transfer their (pension) schemes, mainly driven by changes in accounting standards forcing companies to report their pension liabilities more transparently (Davis, 2008). **Mortality-linked bonds** can be divided into two categories: With *mortality bonds*, investors bear the risk of losing coupon payments and perhaps a part of the principal if an extreme mortality event occurs. The payoff of *longevity bonds* is linked to a pre-defined mortality index; both generally transfer very specific forms of longevity risk. However, mortality-linked bonds have rarely been issued yet. Additionally, they suffer from cross-hedging (or basis) risk as they are based on whole populations and not on the lives within an individual scheme and they do not protect the scheme from stochastic risk (Reed, 2007). **Mortality derivatives:** Tailor-made OTC solutions to transfer the longevity risk component only (Collet-Hirth & Haas, 2007). Recently, Norwich Union has completed its first longevity swap, hedging GBP 475m of exposure (Evans, 2009). Mortality swaps aim to offload the longevity risk whilst retaining asset and investment risk (Reed, 2007). Although longevity swaps have recently gained in popularity, this market is still illiquid and intransparent. The market for longevity futures and options is left even more behind.

STAKEHOLDERS

(Corporate) pension funds, insurance companies and governments (i.e. **hedgers**) need to be able to pass through the risk of unexpected changes in LE, traditionally through reinsurance solutions. Nevertheless, the emergence of mortality-linked bonds has shown that they look for alternatives to spread the risk. Additionally, the hole in many balance sheets has alerted regulators and rating agencies, making further risk management tools such as longevity derivatives essential. However, a well functioning market cannot solely consist of hedgers; one also needs intermediaries, investors, speculators and arbitrageurs. To approach the question of potential market participants, it is important to identify their needs and benefits. Within the private sector, **(corporate) pension funds** are, due to defined-benefit pension liabilities, most exposed to longevity risk; since increasing LE creates additional costs, they are *short longevity*. For **insurance companies** a case differentiation has to be made; those with an overhang of annuity liabilities are *short longevity*, whereas those with a bulge of term life insurances are *long longevity*. Depending on their business mix, they can be flat, long or short on an aggregated basis and the utilization of natural hedges enables them to reduce some of the risk of unexpected changes in mortality rates.

The **government** has an interest in the market for longevity derivatives as hedging reduces the probability of financial distress triggered by a company's pension scheme and thus stabilized the economy; as 'insurer of last resort', the government is also likely to be left with residual claims. Furthermore, the government would be able to manage their own exposure (e.g. in Germany the government provides a significant amount of retirement income). As for other markets, **regulators** have to ensure financial stability through promoting efficient, orderly and fair markets.

The placement of mortality-linked bonds has shown that **investors** are willing to invest in this asset class. They usually seek further diversification since theory suggests that the market portfolio consists of all risky assets; for investors the correlation between traditional asset classes and longevity risk will be key. According to Blake et al. (2006), longevity risk has a low correlation with standard financial market risks, making investments in them attractive. Furthermore, each longevity liability creates an identical asset in someone else's balance sheet and thus the market might be much more balanced than assumed at first glance. **Investment banks** and / or **exchanges** would cover the intermediary function and would benefit from being compensated for providing a market place and market prices through bid-ask spread, transaction fees (exchanges) or sales margins (OTC) and obtaining private information inherent in the flow (microstructure, 'read the flow'). **Speculators** and **arbitrageurs** are likely to appear, especially as immature markets tend to provide arbitrage opportunities due to inefficiencies. They will help to approach the equilibrium price and provide liquidity, where the latter is essential to the success of futures and options markets. The interest of **rating agencies** in the emergence of a market for longevity risk can be attributed to reputation risk. Presently, longevity risk is judged on a model basis, but the recent past (e.g. CDOs) has shown the weakness of marking-to-model approaches. Thus, through the ability of hedging longevity risk, rating accuracy would increase (Blake et al., 2006 and Picone et al., 2008).

The interest rate market has shown that the development of bond and derivative markets go hand in hand as the latter allows achieving the desired risk exposure in an efficient way. Thus, a well functioning market for derivatives appears to be prerequisite to gain further development in mortality-linked bonds as well as to utilize the benefits of market participants. Due to rising concerns of regulators and rating agencies, pension funds and life insurers are likely to be forced to take a more active stance in managing longevity risk, pushing the development of the market for longevity derivatives. Previous innovations have shown that the main pre-requirements for new markets to take off are large and onerous risks that cannot be hedged via existing instruments and the creation of liquidity. Where the first criterion is assumed to be fulfilled, the next step will be to provide a reliable framework to attract liquidity. Recently, the first trades in longevity swaps have been executed, indicating that the need for derivative solutions outweighs the risks inherent in trading premature instruments. However, instruments with non-linear payoffs have not broken through in practice yet although options are an integral part of the derivatives family. Thus, after revising the literature in chapter 2, this paper aims to provide a universal applicable options pricing framework that will be described in chapter 3. Before concluding in chapter 5, the model is numerically applied on German mortality data in chapter 4.

CHAPTER 2:

LITERATURE REVIEW

The literature review summarizes key developments achieved so far. *'The Forward Issue'* will show that forecasting future mortality rates is central when it comes to option pricing. Cairns et al. (2007) outline various approaches that have been proposed for modeling mortality as a stochastic process. Cairns et al. (2008) define three qualitative model requirements, besides sample fit, to generate meaningful forecasts: biological reasonableness of forecast mortality term structures, biological reasonableness of individual stochastic components of the forecasting model and reasonableness of forecasted levels of uncertainty relative to historical levels of uncertainty. Examining all models is beyond the scope of this paper and therefore two models are described more in detail: The Lee-Carter (LC) and the Cairns, Blake and Dowd (CBD) model. Based on US life tables, Lee & Carter (1992) published a one factor stochastic model for central mortality rates which has become the 'leading statistical model of mortality [forecasting] in the demographic literature' (Deaton & Paxson, 2004). Lee (2000) explains that, through regression, (logs of) age-specific death rates can be modeled as a linear function of a periodic-specific index of mortality intensity, with parameters depending on age and time. The model allows for long-run forecasts of age-specific mortality distributions for a desired confidence level since the error term is well-behaved and of small variance. The mortality rate $m(t, age) = e^{a_x + b_x k_t}$ is a function of three parameters; a_x (drift term) describes the overall evolution of a particular age-group (x) over time, b_x indicates the sensitivity of an age-group to time and k_t describes the change in mortality rates over time without differentiation between age-groups. The LC model states that mortality, by and large, increases exponentially in time. The charm of the model is that it is easy to calibrate, given the limited number of parameters and their intuitive meaning (Picone et al. 2008). However, Girosi & King (2007) identify unrecognized and insufficiently appreciated properties of the LC model. They prove that forecasts over long periods violate observed empirical patterns and thus the power of the model is limited. This finding can be attributed to the fundamental weakness of extrapolation that historical patterns may not hold for the future as structural changes may be missed (e.g. medical advances, changing lifestyles, new diseases). Girosi & King also found that the LC model is not applicable to many cause- and country-specific mortality data sets. Thus, over time, more sophisticated models and extensions of the LC model have been developed (e.g. P-splines model; Currie et al., 2004) but the larger number of parameters makes calibration a delicate process. The CBD model[2] (Cairns et al., 2006) is a two-factor stochastic model which has been derived from mortality data for England and Wales. The first factor ($A_1(u)$) affects mortality rate dynamics at all ages in the

[2] Notably, different applications / extensions of the CBD model exist

same way, whereas the second factor ($A_2(u)$) allows for different dynamics between higher and lower ages. They define *forward survival probabilities* as hazard rate $p(t, T_0, T_1, x)$ (i.e. probability as measured at t that an individual aged x at time 0 and still alive at T_0 survives until $T_1 > T_0$). Mortality curves are obtained as follows: $\tilde{q}(t,x) = \frac{e^{A_1(t+1)+A_2(t+1)(x+t)}}{1+e^{A_1(t+1)+A_2(t+1)(x+t)}}$, where $\tilde{q}$ is the *realized* survival probability and $A_1(u)$ and $A_2(u)$ are stochastic processes that are assumed to be measurable at time u. Forecasts are modeled as a two-dimensional random walk with drift. The LC and the CBD model have in common to be estimated from historical data. One of the key differences is that the CBD model smoothens mortality between ages, whereas LC doesn't (note, the Currie model assumes smoothness in both time and age dimension through using P-splines). Based on the sample, the CBD model appears to be robust, especially for higher ages. According to Reed (2007), forecasting mortality is afflicted with three sources of uncertainty: *Modeling risk* (risk that probability distribution is incorrectly modeled), *trend risk* (risk that large unanticipated changes in socio-economic environment or medical advances significantly improve longevity) and *random variation risk* (risk that realized mortality rates vary from expected rates even if probability distribution and trend is modeled correctly).

Cox & Lin (2004) provide evidence that insurer's who utilizes natural hedging charge lower premiums, leading to a competitive advantage. Natural hedge is based on the fact that life insurer liabilities decrease as mortality improves since death benefit payments will be delayed. In contrast, annuity issuers will suffer because they have to pay annuity benefits for a longer time period. They also show that an optimal hedge cannot be achieved through mixing life and annuity risks and propose to introduce mortality swaps. Blake et al. (2006) discuss capital market based solutions to manage systematic longevity risk; they describe the main mortality-linked securities which existed (SwissRe Mortality Bond, issued 2003) or have been announced (EIB/BNP Longevity Bond announced in 2004; withdrawn a year after due to a lack of demand). The former was a 3 year non-capital guaranteed CAT bond allowing to hedge against extreme events, whereas the latter was a 25 year longevity bond; the innovation was to link the coupon to a cohort survivor index based on realized mortality rates. They also outline the use of hypothetical mortality-linked bonds (e.g. survivor bonds, longevity zeros, geared longevity bonds) and derivatives (longevity swaps, options and forwards). Swaps allow interchanging anticipated for actual mortality. Their advantages are low transaction costs, high flexibility and easy cancellation. Futures normally allow trading the underlying risk with low transaction costs and high liquidity; thus the challenge for longevity futures will be to attract liquidity in the spot market first.

Picone et al. (2008) have introduced a model to price longevity swaps. Following the framework provided by interest rate derivatives, longevity swaps are priced through simulating future mortality rates ('forward curve') using the LC approach. They address two categories of longevity swaps. The age-group (cohort) swap hedges the longevity exposure of a particular age-group (cohort). Similar to swaps, practitioners are working on capital market solutions for longevity forwards (OTC) and futures (exchange traded) respectively. Coughlan et al. (2007a) suggests LifeMetrics q-Forwards, where realized mortality rates are exchanged at a future date, in return for a fixed mortality rate agreed at inception (NPV = 0 at inception). The pricing of q-Forwards is similar to interest rate or FX forwards. They expect that q-Forwards will trade below the 'best estimate' mortality rates by assuming that investors require a risk premium to take on longevity risk.

Compared to longevity swaps and futures, literature on longevity options is spread thin. Blake et al. (2006) suggest that the payoff of an option should be linked to a survivor index or futures price and point out the need of good stochastic mortality models (see Cairns et al., 2006). Lin & Cox (2007) propose to link the option's payoff to a population longevity index and they describe the dynamics of the index with a combination of a geometric Brownian motion and a compound Poisson process. They also consider the probability of a jump (i.e. big change in mortality). They come up with a closed-form solution by adjusting the Black-Scholes (BS) formula. In another paper, Cairns et al. (2005) make use of similarities between the force of mortality and short-term risk-free interest rates to introduce arbitrage-free valuation methodologies for the pricing of various mortality-linked instruments by adapting arbitrage-free frameworks that have been developed for interest rate derivatives. They also outline the shortcomings of the arbitrage-free framework if markets are incomplete (i.e. illiquid, not frictionless) and address the need of a non-manipulable, ideally continuously updated mortality index derivatives can be settled against; the main problem of the latter is that mortality data is published much less frequently compared to financial or economic data. JPMorgan (2008) has developed the first public, traded and international longevity index, the LifeMetrics Index. In their paper, Coughlan et al. (2007b) describe LifeMetrics as a toolkit for measuring and managing longevity and mortality risk which is designed to facilitate the structuring of longevity securities and derivatives. Their index is based on publicly available mortality data, broken down by country, age and gender. The components of index calculation are crude central rate of mortality, graduate initial rate of mortality and period LE. JP Morgan tries to overcome the moral hazard problem by an Index Advisory Committee which reviews methodology and data regularly. However, the shortfall of infrequent disclosure is still present as the indices are released annually.

Chapter 3:

Description of the Options Pricing Framework

Mortality rates possess similar characteristics to default rates in the credit world as the event of death is comparable to the bankruptcy of a company. Analogously, increasing mortality rates can be regarded as a kind of financial distress. Where financial ratios (e.g. interest coverage, leverage) are the driving force behind the determination of default probabilities, genetic and non-genetic factors (e.g. health care, lifestyle, social class, geographical location, wealth, education) as well as 'external shocks' affect mortality rates (Blake, 2008). For example, a corporate bond would decrease in value if the creditworthiness weakens and likewise the value of a bond linked to mortality rates would vary depending on changes in mortality rates. Since the financial industry is familiar with the functionality of credit default rates, the option pricing is based on mortality rates and not on survival rates as suggested by other authors. However, having their interrelation ($p_x = 1 - q_x$) in mind, transformation into survival probabilities would be easy.

In this chapter, the underlying is described on the basis of the data set presented in chapter 4. Before the theoretical options pricing framework is introduced, the forward issue is discussed in detail as forecasting future mortality rates is central when it comes to the valuation of longevity options.

In figure 1, mortality rates q_x for 0 – 89 aged German males are plotted for the years 1957, 1965, 1975, 1985, 1995 and 2005. It shows that they are not normally distributed along ages but have a fat left tail (high infant mortality), a relatively flat core and exponentially increasing mortality rates for older ages. The described structure of mortality rates is consistent with the literature (e.g. Sweeting, 2008). Thus, in a first step a theoretical distribution has to be identified that fits empirical data (see '*Fitting the spot distribution*').

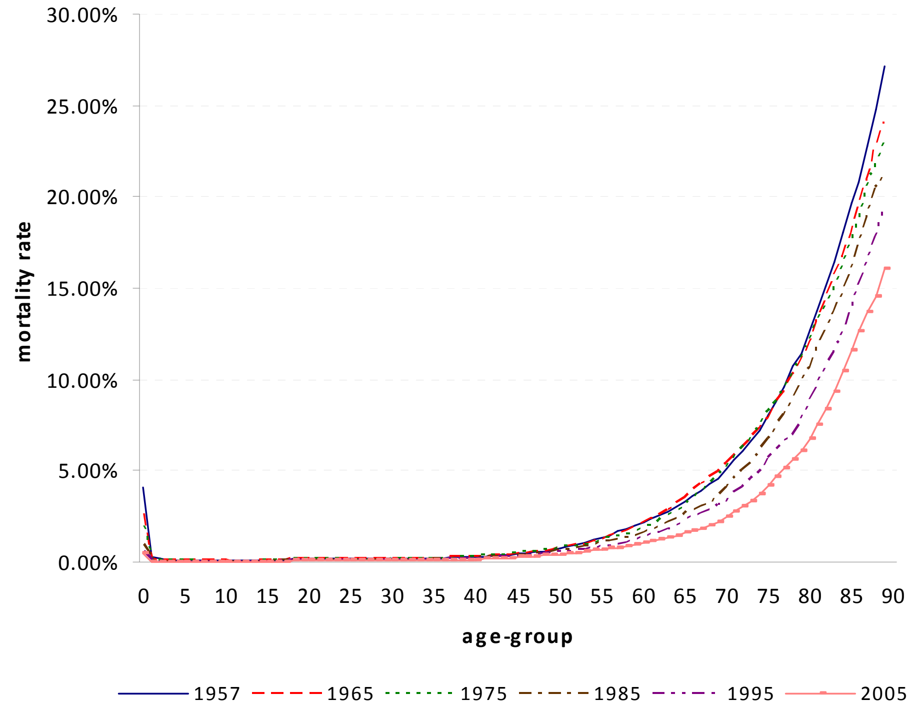

Figure 1. Mortality distributions for German males (1957 – 2005)

Figure 2 illustrates the trend of decreasing mortality rates for different age-groups (0, 20, 40, 60, 80; all German males) as time passes by. It can be observed that mortality rates of both infants and old aged males significantly decrease over time (infants: 4.06% in 1957 vs. 0.44% in 2005; 80 year old male: 12.77% in 1957 vs. 6.78% in 2005), whereas mortality rates of other age-groups are relatively flat. These patterns are mainly due to two reasons: the base effect and mortality improvements. The former (less slope for age-groups in-between the 'extreme' left and right tail) is due to the fact that the mortality rate of say 10 year old males was already just 0.05% in 1957 (vs. 0.01% in 2005) and thus decreasing mortality rates are less significant in absolute terms; the latter on the other hand can be attributed to factors such as advances in health care, improving living conditions, increasing wealth etc.. For example, heart disease improvements have become increasingly significant since the mid-1970s. As elder people are more likely to suffer from heart disease, medical advances impact age-groups differently. This example shows that those trends are firstly not static (e.g. age-group 80: sideward trend till mid 1970s, afterwards downward trended) and secondly are different across age-groups. To outline potential variability of the trend, one might imagine the impact of the emergence of drugs to cure HIV, especially on the mortality rates of African countries. This result is consistent with common view; Cairns et al. (2006) describe the downward trend in mortality rates and present evidence that mortality improvements are stochastic as the rate of improvement has varied significantly, and improvements have varied substantially between age-groups so that the general trend comprises an unpredictable element. The resulting estimation error is due to the nature of extrapolation and forecasting accuracy is likely to be a function of time.

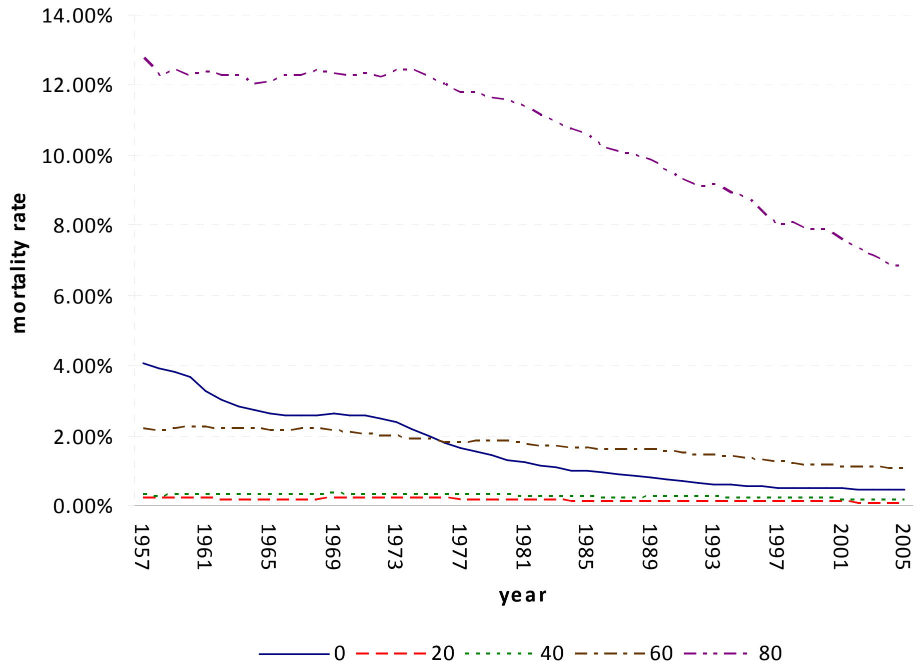

Figure 2. Mortality rates for different age-groups (1957-2005)

Although not included in the data set analyzed, it is reasonable to assume jumps in mortality rates in the case of exceptional circumstances such as wars or pandemics. Those events have a small probability to occur but potentially have a significant impact on the payoff of longevity options and thus must be factored in the pricing. It is also appropriate to assume mean reversion, meaning that mortality rates will return to their path as soon as those exceptional circumstances disappear (either instantaneously or lagged). Furthermore, one could also consider slope dummies to model mortality improvements that would instantaneously cause significantly mortality improvements. Again, those events certainly have a low probability but mortality rates would drop instantaneously (no mean reversion) and potentially strongly influence the options value. However, the issue with jumps and slope dummies is that, due to a lack of empirical data, they are building on subjective assumptions.

To sum up, the development of the underlying can be decomposed into four components: An underlying spot *distribution* (current mortality rates), a *trend* (mortality improvements), a *jump* to allow for extraordinary circumstances and a *slope dummy* to model instantaneous and significant mortality improvements.

THE FORWARD ISSUE

Generally, longevity options can be priced by applying arbitrage-free pricing frameworks for interest rate derivatives. Where forward interest rates are the rates of interest implied by current zero rates for periods of time in the future, such an arbitrage relationship between the spot and the forward market does not exist for mortality rates. Therefore, one main difference between interest rate and longevity options is the way the forward curves are determined (Picone et al., 2008).

The standard market model for interest rate caps and floors (Black-76 model; B76) is an extension of the BS model. A European call (cap) is priced as follows: $c = e^{-rT}[FN(d_1) - KN(d_2)]$ where $d_1 = \frac{\ln(F/K) + (\sigma^2/2) \times T}{\sigma \times \sqrt{T}}$ and $d_2 = d_1 - \sigma \times \sqrt{T}$; the formula for a European put (floor) is $p = e^{-rT}[KN(-d_2) - FN(-d_1)]$ accordingly (Hull, 2009). Compared to the elementary form, the only difference is that the forward replaces the spot price. In the interest world, different hypothesis exist to explain the term structure of interest rates and thus the forward curve (unbiased expectations hypothesis, liquidity premium hypothesis, preferred habitat hypothesis and market segmentation hypothesis), each able to partially explain the term structure but some of them are mutually exclusive and overall empirical evidence is relatively weak (Hull, 2009). Nevertheless, due to the existence of forward rate agreements (FRAs), an arbitrage relationship between the spot and the forward market exists as a spot investment for say 1 year must yield the same as an investment for 6M spot and a covered position in 6M for 6M.

Such an arbitrage relationship does not exist for mortality rates and thus the forward curve for mortality rates is obtained through simulation, so that the valuation of longevity options requires good stochastic mortality models (Blake et al., 2006). However, both the *'description of the underlying'* and the *'literature review'* highlighted issues surrounding the predictability of future mortality rates and no model exists that fits all case- and country-specific data sets best (e.g. LC has a good sample fit for US data, whereas CBD gets robust results for England and Wales); this in conjunction with uncertainty about future mortality rates makes it reasonable to assume that the simulated forward is a good proxy for future spot rates at best. Since various models exist to forecast future mortality rates and models can be adjusted for expectations (e.g. assumptions regarding the probability of jumps or the coefficients of the trend), the simulated forward can be regarded as *'unbiased expectation'*, meaning that the forward is the best estimate for the future spot rate. Obviously, supply-demand issues, as suggested by the market segmentation hypothesis, might cause deviation from the simulated forward. This fact in conjunction with the assumption that diverse market participants estimate future mortality rates based on different models and input parameters respectively is likely to create a range of forwards, reflecting the uncertainty of extrapolation and the absence of a spot-forward arbitrage relationship. As long as the forward remains within the boundaries, arbitrage is unlikely to appear; however, as the forward departs too far from its 'fair value', arbitrageurs will appear and push the price back into the range.

Forward mortality rates are country-, gender- and age-group specific. Where interest rates have one forward curve for each currency, the market for longevity derivatives will be based on two forward surfaces per region (males and females). Notably, country specific forward curves might not be accurate enough as LE can significantly differ within one country. Exemplarily, the LE for males (age-group 0) in *Mecklenburg-Western Pomerania* is 74.85 years compared to 78.33 years in *Baden-Wuerttemberg.* However, this issue will not be discussed in detail.

To sum up, like interest rate forwards, mortality forward rates will not be a perfect predictor for future spot rates, but simulated forwards can be applied to price longevity options under arbitrage-free interest rate frameworks. Plenty of suggestions exist how to simulate the forward; due to the application of different methods and varying expectations amongst market participants, it is likely to get a range of forward mortality rates. Market forces will ensure that forward rates will remain within this range and to approach the equilibrium price.

THE MODEL

The option pricing framework presented in this subsection claims to be universally applicable. Hence, an approach is introduced that easily allows calibrating for country- and case specific factors as well as individual expectations regarding future mortality rates. The option pricing framework is based on age-groups but can easily be extended to cohorts. Mortality rates are chosen as strike; for practical applications this implies that, say a corporate pension fund, has to estimate its sensitivity to unexpected changes in mortality rates for a certain age-group and can hedge it's exposure by choosing strike and notional value accordingly. Example: Suppose an annuity provider (short longevity / long mortality rates) expects the mortality rate for the age-group '*Germany, males, 80*' to decline from 7.00% to 6.50% over the next 5 years and has priced its annuity products on this projection. They face the risk that mortality rates might drop below the forecasted value and have evaluated that a decline of 10 bps would create additional costs of GBP 100,000 for this age-group. They could hedge themselves by purchasing a put with strike 6.50%, 5 years time to maturity and a notional value of GBP 100m as this option would offset the risk of unexpected declines in mortality rates whilst maintaining the upside potential of increasing mortality rates.

To value options, the underlying stochastic process has to be modeled; it consists of a spot *distribution*, a *trend* and *jumps* (i.e. Lévy process; a Lévy process contains three components: a drift, a diffusion and a jump component (Applebaum, 2004)). Changes in mortality rates, can be modeled as follows: $dX = f \times dt + \sigma \times W + g \times J$, where dX = change in mortality, f = trend, dt = change in time, σ = volatility, W = random walk, g = jump size and J = process that counts jumps. Generally, hazard functions (or hazard rate or conditional failure rate) describe the conditional probability that a certain event (e.g. death) occurs within the interval $[t, t + \Delta t]$ under the condition that the event has not occurred until t. They are applied to answer questions such as "*how long do humans live?*". Hazard functions assume that the underlying follows a certain process where the probability of the event increases or decreases depending on the factor time (Mortensen, 2005). The *Weibull* and the *Gompertz* distribution (Milevsky & Promislow, 2001) can be regarded as subsets of hazard functions and both are commonly used in survival analysis and can be applied to fit the sample data to obtain the underlying distribution (see use of Weibull in '*Fitting the Spot Distribution*'). Nevertheless, since the forward is determined on the basis of a given spot distribution, this exercise is less crucial compared to modeling the trend. As mentioned earlier, the impact of mortality improvements affects various age-groups differently. Thus, to determine the trend, regression analyses are performed for each age-group. Without considering jumps or dummies yet, forecasts for future mortality rates are obtained by adjusting the spot distribution for the projected trend. As outlined earlier, mortality rates might not change continuously but jump in case of extraordinary circumstances; it can also be assumed that mortality rates will be pulled back to their predicted path as soon as those external shocks disappear (i.e. mean reversion). To model jumps, assumptions regarding their probability, amplitude and duration have to be made, usually by analyzing the empirical distribution. The jumps are notably one-sided so as to capture the effect of increasing mortality rates. Similarly, significant mortality improvements can be captured by slope dummies. Therefore, assumptions regarding their probability and their age-group specific effects have to be made.

To sum up, the continuous process as described above is likely to be overlaid with mean reverting jumps as well as instantaneous but non mean reverting mortality improvements. The proposed option pricing framework can therefore be considered to be a mixed jump-diffusion model (Hull, 2009).

As mentioned earlier, longevity options can be priced under the arbitrage-free (or risk-neutral) condition. As a consequence of risk neutrality (i.e. all individuals are indifferent to risk and thus do not require a premium for bearing risks), the option value is obtained by discounting future cash flows at the risk-free rate of return. Risk-neutral valuation states that a risk-neutral world can be assumed when pricing options and the resulting option values are correct, not only in a risk-neutral world. In the simplest form, the underlying is assumed to behave binomially, meaning that the underlying either goes up or down. Then, portfolios with identical payoffs in either state are created (portfolio replication) so that the future cash flow is known with certainty and thus allows the use of arbitrage-free pricing. In other words, by continuously delta hedging, the risk-free rate can be extracted. E.g., a short call can be hedged by being long Δ the underlying as the cash flows would theoretically offset each other and thus the risk-free rate of return will be earned; notably, Δ is time varying so that the hedge is of a dynamic nature. However, issues surrounding delta hedging, especially when the underlying process contains jumps, are not part of this paper (Hull, 2009). In the case of longevity options, the underlying, human lives of a certain age-group, are a non-tradable asset. However, recalling interest rate derivatives, caplets and floorlets are essentially options on the forward interest rate (see B76 formula) so that they can be hedged with appropriate positions in the LIBOR forward market, most commonly by using futures contracts due to their high liquidity (Gupta & Subrahmanyam, 2005). Likewise, longevity options are priced on the basis of forward mortality rates and thus can be hedged by futures contracts. The LifeMetrics q-Forwards framework seems to be appropriate for hedging purposes since it references to the same input parameters like the option framework proposed (i.e. age-group, mortality rate, gender, and country). However, the main issue will be to attract liquidity in the q-Forwards /q-Futures market to allow for (cost) effective delta hedging as the application of arbitrage-free methods is problematic in incomplete markets. However, *if* a liquid market for longevity futures evolves, *then* the model will be arbitrage free (Cairns et al., 2005). Due to the needs of a market for longevity derivatives, it is likely that futures contracts will become liquid in the near future. To sum up, it has been shown that longevity options can be delta hedged and thus the concept of risk-neutral valuation is valid.

Another principal that has to hold is *put-call-parity*. Dawson et al. (2008) have observed that this condition is independent of the price distribution. In chapter 4 it will be examined whether or not the relationship between European put (p) and call (c) prices with identical strike K and time to maturity T holds, so that $p = c - e^{-r \times T}(F - K)$, where F = forward price and r = risk-free rate of return.

Generally, three main families of option pricing tools exist: closed- form solutions, with the BS model as its most famous application and numerical procedures such as tree models and Monte Carlo simulations (MCS). The charm of BS in its basic forms, besides it's simple pricing formulas, is that sensitivities can be mathematically derived through partial differential equations so that these results can be regarded being very precise. On the other hand, the model suffers from various assumptions that have to be made. One of these assumptions is that the probability distribution of the underlying at any given future time is lognormal, which is clearly not true for mortality rates. To allow for non-lognormality, the volatility smile is a commonly used tool to calibrate the model. Additionally, for complex underlying processes and / or multiple input factors, analytical solutions might not readily be available so that approximation techniques have to be implemented. Tree models are usually applied to price options where the holder has decisions to make prior to maturity (e.g. American options).

MCS enable to value options with multiple sources of uncertainty, complex underlying processes or with complicated features and appear to be very flexible (Hull, 2009). In a MCS, multiple paths that describe the underlying process are created and the average payoff is present valued to obtain the value of an option. As MCSs are relatively time-consuming and thus slow, they are usually only applied where analytical solutions do not exist. Nevertheless, this approach seems to fit the requirements of longevity options best as they allow capturing the underlying process of mortality rates as well as multiple sources of uncertainty. Thus, the numerical application in chapter 4 is based on a MCS with the following input parameters:

- Simulated forward $F_{(Country,\ Gender,\ Age\text{-}group)}$
- Volatility $\sigma_{(Country,\ Gender,\ Age\text{-}group)}$
- Time to maturity T
- Risk-free interest rate r
- Strike K

CHAPTER 4:

NUMERICAL APPLICATION

This chapter consists of five subsections. Firstly, the data is introduced before backtesting the proposed forward determination. Thirdly, it is shown how to fit the spot distribution by applying the Weibull method. Afterwards, options are priced on the basis of the data set. Finally, a sensitivity analysis for both the 'greeks' and other input parameters is undertaken.

THE DATA

To derive option prices and test the model, mortality data provided by the *Statistisches Bundesamt, Wiesbaden* (Germany) are applied. The analysis rests upon mortality tables for West Germany for the period from 1957 to 2005 for age-groups between 0 and 89 years; the analysis is performed for males and females. Unfortunately, mortality data for World War I (1914 – 1918) and II (1939 – 1945) are not available. Hence, jumps cannot be modeled on the basis of empirical distributions but solely on assumptions.

BACKTESTING

The backtesting evaluates the accuracy of the simulated forward. Since the data set does not contain 'external shocks', the evaluation will be based on the respective spot distribution and a trend. The following methodology is applied: The spot distribution is defined as the most recently published mortality rates ($t = 0$). The results of linear ($y = a \times x + b$), logarithmic ($y = a \times \ln(x) + b$) and exponential ($y = b \times e^{a \times x}$) trends are compared against each other. Intuitively, logarithmic or exponential trends seem to be appropriate since both decreases at a decreasing rate; the problem with linear trends is that they might not be reasonably logical (e.g. negative mortality rates for long term forecasts). The trend for each age-group is derived from the previous 10 years ($t = -10,...,-1$). To forecast the forthcoming 10 years ($t = 1,...,10$), the spot distribution is extrapolated with the trend by assuming that mortality improvements will possess similar characteristics. Alternatively, the general trend can be obtained through regression and age-group specific behavior can be expressed by different correlation coefficients. The forecasting error is defined as the absolute difference between realized and forecasted mortality rates in $t = 1,...,10$. The average error expresses the average forecasting error across time and age-groups. Backtestings will be performed for both males and females on a rolling basis for the years 1967 – 1996 (i.e. 30 (1967 – 1996) * 3 (linear / logarithmic / linear trend) * 2 (males / females) = 180 tests); for example, the *backtest 1967* is build upon data from 1957 – 1966 to forecast mortality rates for 1967 – 1976. The results for males and females are summarized in table 1 and 2 respectively (details see Appendix B).

Backtest (male)	Linear (average error)	Logarithmic (average error)	Exponential (average error)
Average error	0.1538%	0.2423%	0.1511%
Min error	0.0765% (year 1994)	0.1148% (year 1967)	0.0692% (year 1993)
Max error	0.2836% (year 1972)	0.3148% (year 1974)	0.2893% (year 1972)

Table 1. Backtesting males (1967 – 1996)

Backtest (female)	Linear (average error)	Logarithmic (average error)	Exponential (average error)
Average error	0.1033%	0.1967%	0.0899%
Min error	0.0385% (year 1978)	0.0851% (year 1967)	0.0331% (year 1981)
Max error	0.2418% (year 1972)	0.3349% (year 1974)	0.2438% (year 1972)

Table 2. Backtesting males (1967 – 1996)

Unexpectedly, forecasting with logarithmic trends delivers the worst result as the rate of decrease of mortality rates appears to be too low. **Exponential trends** should be applied to forecast mortality rates since they are firstly logical reasonable and secondly slightly more accurate than linear trends. Figure 3 displays forecasted vs. realized mortality rates across different age-groups (*backtest 1981, females*). Notably, this is the most accurate forecast; best-, average- and worst- forecasts for both males and females are exhibited in Appendix C.

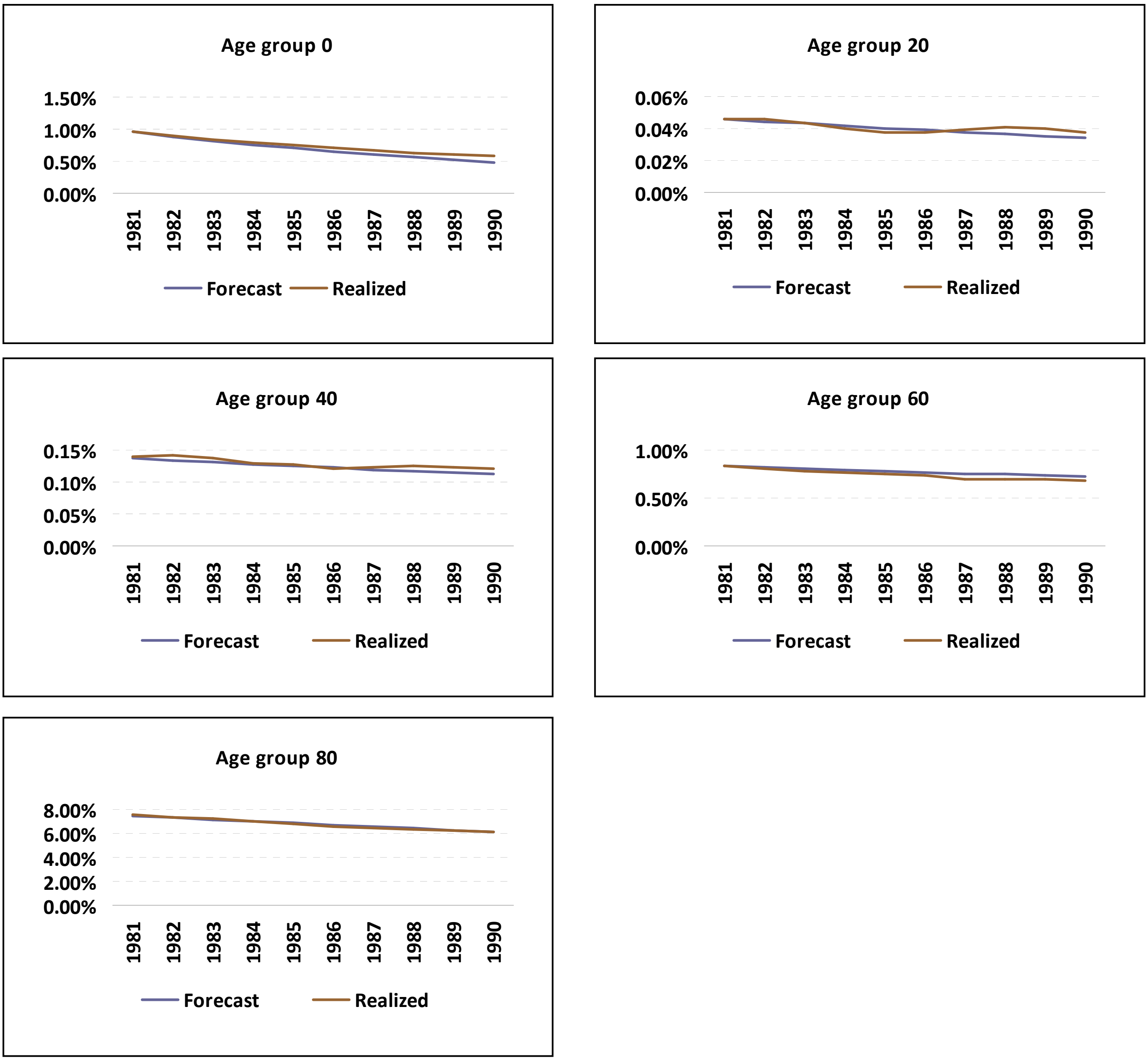

Figure 3. Forecasted vs. realized mortality rates

These findings confirm that mortality rates are difficult to forecast. Although the prediction turns out to be relatively precise in many backtestings, extrapolation delivers poor results in the case of significant trend changes (e.g. 1970s due to significant medical advances). Additionally, forecasting accuracy decreases as a function of time. In practice, the volatility surface accounts for both estimation errors and uncertainty over time by charging a volatility premium for decreasing confidence. However, the volatility surface is not the subject of this paper but a potential area for further research as pricing options is often referred to as the pricing of volatility. Analogue, pricing longevity options means pricing volatility and the forward, whereas this paper focuses on the latter.

FITTING THE SPOT DISTRIBUTION

The Weibull methodology is applied to fit the spot distribution; this is not of particular importance for the suggested option pricing as it is build on the extrapolation of given spot values. Nevertheless, a mathematical description might be of interest to find a closed-form solution as this would increase accuracy and be less time consuming.

The Weibull cdf is commonly displayed as $\hat{F}(t) = 1 - e^{-\left(\frac{t}{T}\right)^{b}}$, where t = age-group and $\hat{F}(t)$ = estimated mortality rate (vs. *F(t)* = actual mortality rate). By linearizing the cdf ($y = \ln\ln\frac{1}{1-\hat{F}(t)} = b\ln(t) - b\ln(T)$), where $T = \exp^{-\left(\frac{a}{b}\right)}$ ('characteristic lifetime'), estimates for b (slope), a (intercept) and thus T are obtained through regression (www.weibull.com, 2009). To get a good fit, the spot distribution has to be decomposed into small intervals (e.g. age-group [0;4], [5;9], …, [85;89]) and a as well as b have to be estimated for each interval. Notably, the value for *F(0)* is obtained via $\lim_{t\to 0}$. The fitting error is defined as $\hat{F}(t) - F(t)$. The results for the 2005 spot distributions are exhibited in figure 4 and 5 (details see Appendix D).

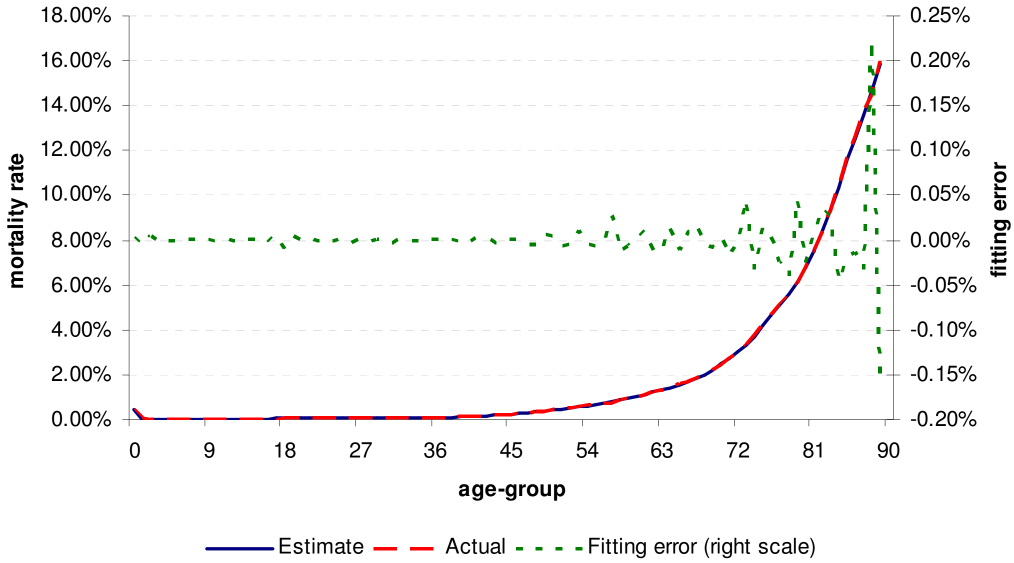

Figure 4. Weibull fit, males (2005)

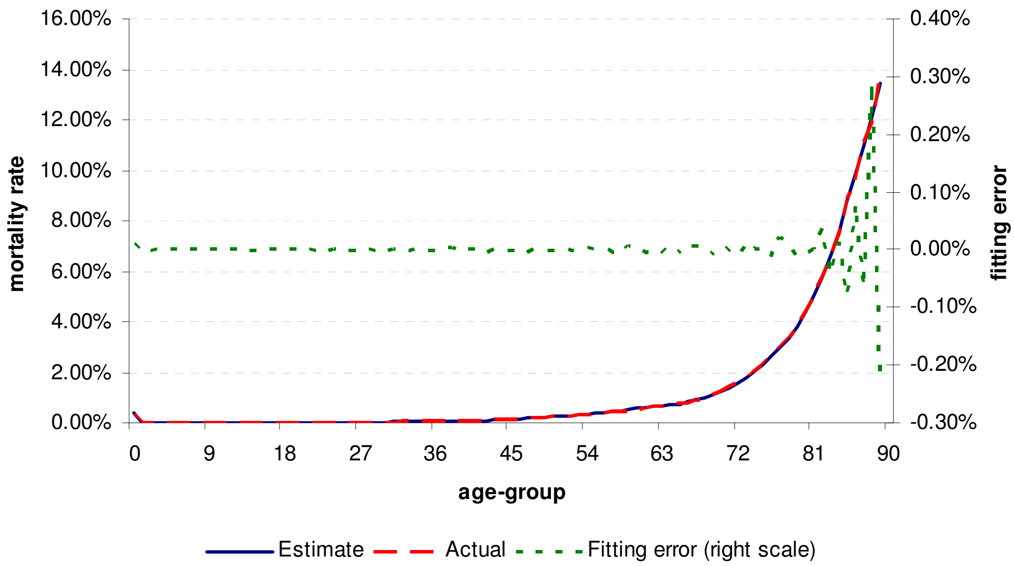

Figure 5. Weibull fit, females (2005)

OPTION PRICING

European puts and calls for males as well as females are priced for the age-groups 80, 60, 40, 20 and 0. In scenario 1, (forward) at-the-money (ATM), 10% (forward) in-the-money (ITM) and 10% (forward) out-of-the-money (OTM) options are priced for 10 years, 5 years and 6 month time to maturity; the strikes as determined in scenario 1 are applied in scenario 2 and 3 to be able to interpret the results. The risk-free rate is assumed to be 5.00% p.a. (flat yield curve). Option prices are calculated for three cases:

1. scenario: Forward determination on the basis of the spot distribution and age-group specific trends
2. scenario: Scenario 1 is augmented to allow for mean reverting jumps to model 'shocks'
3. scenario: Scenario 2 is augmented to allow for slope dummies to model significant and instantaneous mortality improvements

SCENARIO 1

At first, forward mortality rates are simulated by applying the concept introduced in the '*Backtesting*' before volatility parameters are estimated. The forward mortality rates surfaces are summarized in figure 6 and 7 (details see Appendix E);

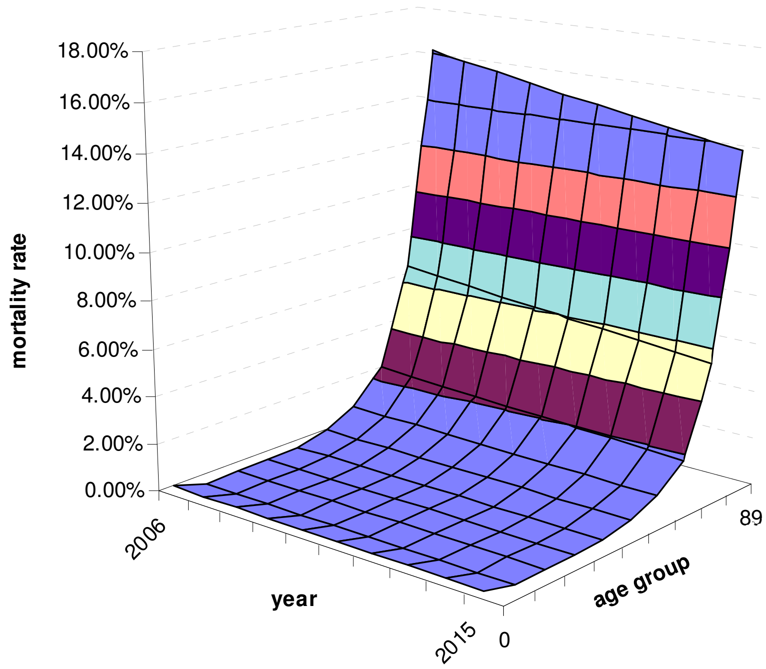

Figure 6. Mortality rates surface, males (2006 – 2015)

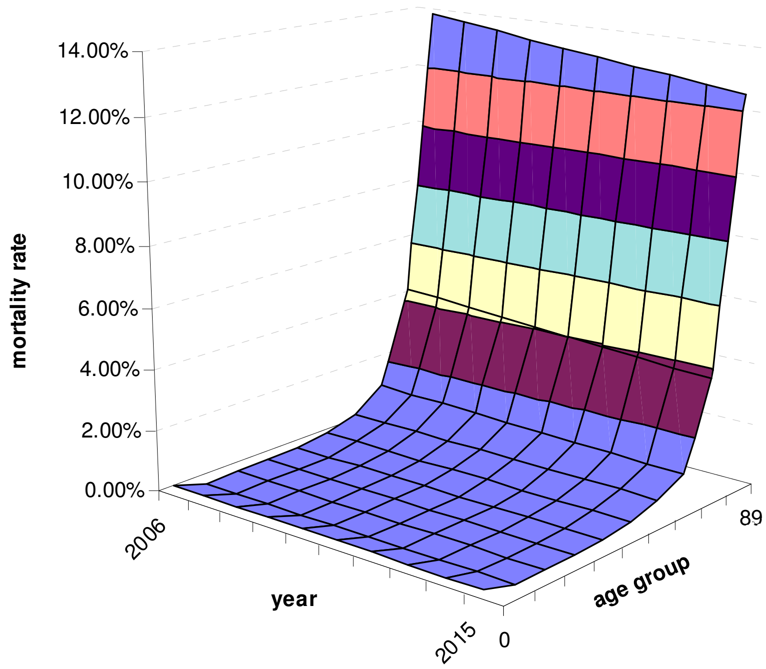

Figure 7. Mortality rates surface, females (2006 – 2015)

As mentioned earlier, the volatility surface is not the subject of this paper. Thus, for simplicity, gender and age-group specific historical volatilities are estimated as standard deviation of logarithmic changes in mortality rates from 1957 – 2005. This means that options are priced with flat volatility, knowing that usually volatility premiums are charged for longer maturities. The results are exhibited in figure 8.

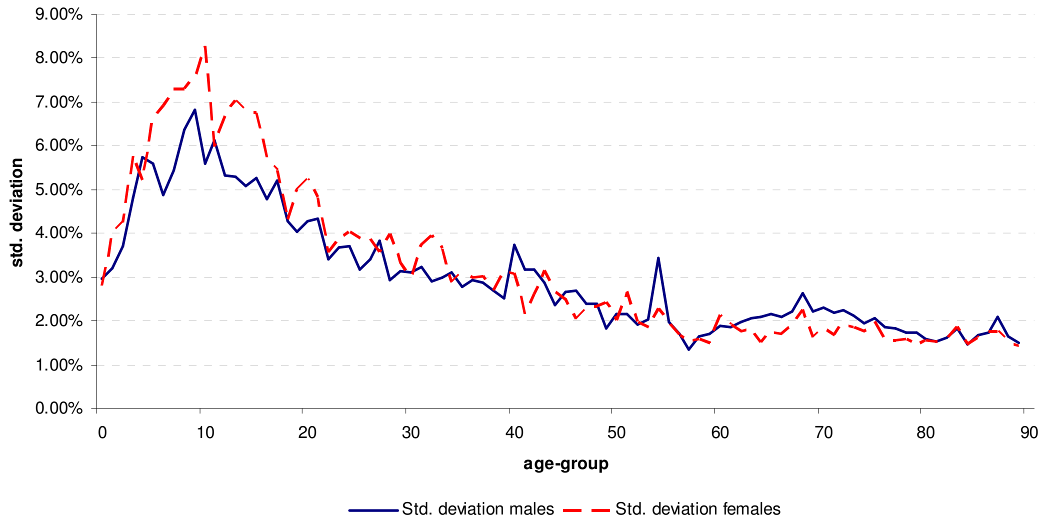

Figure 8. Historical volatilities

For the MCS it is assumed that mortality rates follow a trend. To simulate a path, the Euler method is applied: $p_{(t+1,x)} = p_{(t,x)} \times \exp\left[(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}\phi\right]$, where $p_{(t,x)}$ = mortality rate at time t for age-group x, σ = volatility p.a., δt = time step (e.g. 1/365 for daily steps), ϕ = normally distributed random errors and μ = drift rate p.a. The drift is the sum of the risk-free rate and the age-group specific (negative) slope as determined through regression (i.e. disregarding the risk-free rate and stochastic, $p_{(T,x)}$ equals the forecasted mortality rate).

The option value is calculated as present value of the average expected payoff *E[payoff]* at maturity (i.e. $option\,value = E[payoff] \times e^{-(r \times T)}$), with $payoff_{Call} = \max(p_{(T,x)} - K, 0)$ and $payoff_{Put} = \max(K - p_{(T,x)}, 0)$. To calculate the average expected payoff, a large number of random walks (see figure 9 for 3 random paths; *age-group 20, females, historical volatility 5.22%*

p.a., risk-free rate 0% p.a.) is simulated and the payoffs of each path are averaged.

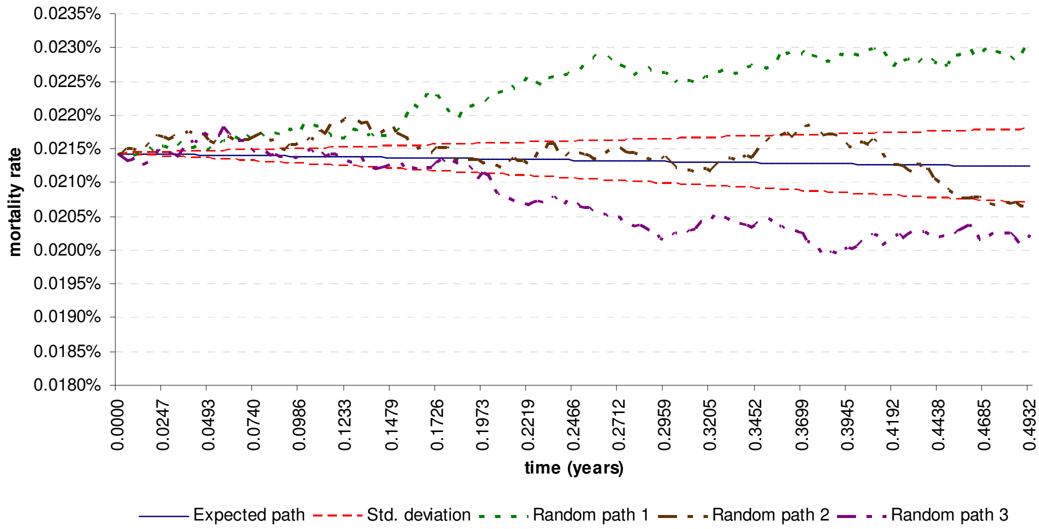

Figure 9. Random walks

The pricing methodology in excel is explained by means of the example *males, age-group 80, 10 years time to maturity, strike ATM* (see figure 10). Excel area A1:C9 shows the simulation input; based on monthly time steps, 1,000 random paths are generated (A11:DR1012). In DS13:DS1012 (DT13:DT1013), call (put) payoffs are calculated for each path. DS1013 (DT1013) shows the average call (put) payoff; the call (put) value in DS1014 (DT1014) equals the present value of the average payoffs; the estimated call (put) price is 0.1079% (0.1064%).

	A	B	C	D	DP	DQ	DR	DS	DT	DU	DV
1	**Simulation Input**										
2	age-group	80									
3	slope	-2.28%	p.a.								
4	r	5.00%	p.a.								
5	drift	2.72%	p.a.								
6	K	8.90%									
7	σ	1.60%	p.a.								
8	T	10	years								
9	Time steps	120	monthly								
10											
11		**Step 0**	**Step 1**	**Step 2**	**Step 118**	**Step 119**	**Step 120**	**Call Payoff**	**Put Payoff**		
12		**0.00**	**0.08**	**0.17**	**9.83**	**9.92**	**10.00**				
13	**Sim 1**	6.7833%	6.7935%	6.7699%	9.6235%	9.6712%	9.6869%	0.7867%	0.0000%		
14	**Sim 2**	6.7833%	6.8018%	6.8576%	8.3137%	8.3452%	8.4348%	0.0000%	0.4655%		
15	**Sim 3**	6.7833%	6.7530%	6.7578%	8.5019%	8.5198%	8.5972%	0.0000%	0.3030%		
1010	**Sim 998**	6.7833%	6.8037%	6.8593%	8.7752%	8.7056%	8.7219%	0.0000%	0.1783%		
1011	**Sim 999**	6.7833%	6.8042%	6.7987%	8.8310%	8.7600%	8.8654%	0.0000%	0.0349%		
1012	**Sim 1000**	6.7833%	6.8167%	6.8216%	8.8131%	8.8053%	8.9027%	0.0025%	0.0000%		
1013							**Average**	0.1779%	0.1755%		
1014							**PV**	0.1079%	0.1064%		
1015							**Stdev**	0.1622%	0.1583%		
1016							**Call 95% CI**	0.0978%	0.1179%		
1017							**Put 95% CI**	0.0966%	0.1162%		
1018											
1019							**Put price (PC parity)**		0.1079%	**PC HOLDS**	

Figure 10. Option pricing in excel

The standard deviation of the 1,000 option values in DS1015 (call) and DT1015 (put) respectively is required to calculate the upper and lower boundary of the option value for a given confidence level, and to determine the standard error; the standard deviation for the call is 0.1622% and 0.1583% for the put. With 95% confidence (i.e. 1.96 numbers of standard deviation) the call value will be between $0.1079\% - \frac{1.96 \times 0.1622\%}{\sqrt{1000}} = 0.0978\%$ (DS1016) and $0.1079\% + \frac{1.96 \times 0.1622\%}{\sqrt{1000}} = 0.1179\%$ (DT1016). The put value should be in-between 0.0966% (DS1017) and 0.1162% (DT1017) accordingly. DT1019 and DU1019 show that put-call parity holds for the given confidence level.

At this stage no judgment about the accuracy can be made. As the number n of random walks increases, the pricing precision increases as the standard error decreases. The standard error is defined as $\frac{s_c}{\sqrt{n}}$ where s_c = standard deviation of the option values. The error is of order $\frac{1}{\sqrt{n}}$, meaning that four times as many simulations have to be run to divide the error by two (see figure 11 for example above).

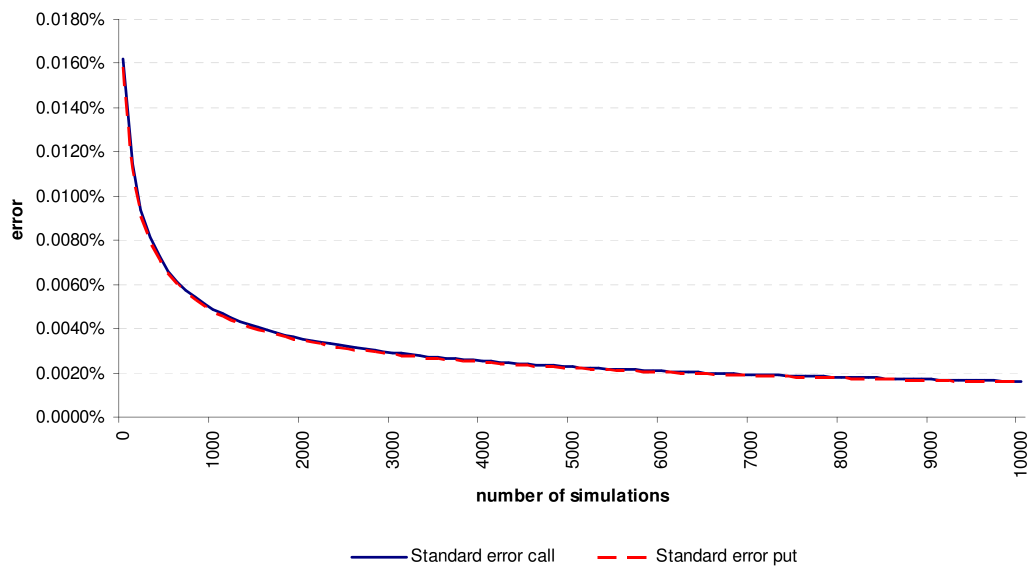

Figure 11. MCS error

Since scenario 1 is a pure drift model, MCS results should converge against B76 (formula see *'The Forward Issue'*) values[3] as n increases. Therefore, it is a good benchmark the results of scenario 2 and 3 can be compared against. The results for the option values are summarized in table 3 and 4.

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[3] Black-76 values in table 3 and 4 are calculated with a pricer provided by the Columbia University, (http://www.math.columbia.edu/~smirnov/normodel.xls)

Time to maturity	Strike	Call (MCS)	Call (B76)	Put (MCS)	Put (B76)	Put (PC Parity)
Age-group 80, male, volatility 1.60% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	8.9002%[4]	0.1079%	0.1086%	0.1064%	0.1086%	0.1079%
	9.7902%[5]	0.0038%	0.0032%	0.5422%	0.5431%	0.5436%
	8.0102%[6]	0.5438%	0.5415%	0.0026%	0.0017%	0.0040%
5 years	7.7700%	0.0843%	0.0861%	0.0846%	0.0861%	0.0843%
	8.5470%	0.0001%	0.0003%	0.6055%	0.6054%	0.6053%
	6.9930%	0.6055%	0.6052%	0.0002%	0.0001%	0.0001%
6 month	6.8747%	0.0285%	0.0300%	0.0303%	0.0300%	0.0285%
	7.5622%	0.0000%	0.0000%	0.6725%	0.6707%	0.6707%
	6.1873%	0.6689%	0.6707%	0.0000%	0.0000%	0.0000%
Age-group 60, male, volatility 1.89% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	1.3723%	0.0197%	0.0198%	0.0194%	0.0198%	0.0197%
	1.5095%	0.0013%	0.0012%	0.0843%	0.0845%	0.0845%
	1.2351%	0.0844%	0.0840%	0.0009%	0.0007%	0.0012%
5 years	1.1930%	0.0153%	0.0157%	0.0154%	0.0157%	0.0153%
	1.3123%	0.0001%	0.0002%	0.0931%	0.0931%	0.0930%
	1.0737%	0.0930%	0.0930%	0.0001%	0.0001%	0.0001%
6 month	1.0515%	0.0052%	0.0054%	0.0055%	0.0054%	0.0052%
	1.1567%	0.0000%	0.0000%	0.1029%	0.1026%	0.1026%
	0.9464%	0.1023%	0.1026%	0.0000%	0.0000%	0.0000%
Age-group 40, male, volatility 3.74% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	0.1463%	0.0042%	0.0042%	0.0041%	0.0042%	0.0042%
	0.1609%	0.0012%	0.0013%	0.0101%	0.0102%	0.0101%
	0.1317%	0.0100%	0.0099%	0.0011%	0.0010%	0.0011%
5 years	0.1419%	0.0036%	0.0037%	0.0036%	0.0037%	0.0036%
	0.1560%	0.0006%	0.0006%	0.0117%	0.0117%	0.0117%
	0.1277%	0.0115%	0.0115%	0.0004%	0.0004%	0.0004%
6 month	0.1380%	0.0013%	0.0014%	0.0014%	0.0014%	0.0013%
	0.1518%	0.0000%	0.0000%	0.0135%	0.0135%	0.0135%
	0.1242%	0.0134%	0.0135%	0.0000%	0.0000%	0.0000%
Age-group 20, male, volatility 4.26% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	0.0537%	0.0017%	0.0017%	0.0017%	0.0017%	0.0017%
	0.0590%	0.0006%	0.0006%	0.0038%	0.0039%	0.0039%
	0.0483%	0.0038%	0.0038%	0.0005%	0.0005%	0.0005%

[4] ATM (i.e. $p_{(t=0,x)} \times e^{(drift \times time-to-maturity)}$)

[5] ATM * 1.1

[6] ATM * 0.9

5 years	0.0569%	0.0016%	0.0017%	0.0017%	0.0017%	0.0016%
	0.0626%	0.0004%	0.0004%	0.0048%	0.0048%	0.0048%
	0.0512%	0.0047%	0.0047%	0.0003%	0.0003%	0.0003%
6 month	0.0599%	0.0007%	0.0007%	0.0007%	0.0007%	0.0007%
	0.0659%	0.0000%	0.0000%	0.0059%	0.0058%	0.0058%
	0.0539%	0.0058%	0.0058%	0.0000%	0.0000%	0.0000%
Age-group 0, male, volatility 2.96% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	0.5903%	0.0132%	0.0133%	0.0131%	0.0133%	0.0132%
	0.6493%	0.0027%	0.0028%	0.0383%	0.0386%	0.0385%
	0.5313%	0.0382%	0.0379%	0.0022%	0.0021%	0.0024%
5 years	0.5117%	0.0103%	0.0105%	0.0103%	0.0105%	0.0103%
	0.5629%	0.0009%	0.0009%	0.0408%	0.0408%	0.0408%
	0.4605%	0.0405%	0.0404%	0.0007%	0.0006%	0.0006%
6 month	0.4499%	0.0035%	0.0036%	0.0037%	0.0036%	0.0035%
	0.4949%	0.0000%	0.0000%	0.0441%	0.0439%	0.0439%
	0.4049%	0.0437%	0.0439%	0.0000%	0.0000%	0.0000%

Table 3. Scenario 1 option premiums, males

Time to maturity	Strike	Call (MCS)	Call (B76)	Put (MCS)	Put (B76)	Put (PC Parity)
Age-group 80, female, volatility 1.55% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	6.0288%	0.0708%	0.0713%	0.0698%	0.0713%	0.0708%
	6.6317%	0.0022%	0.0018%	0.3669%	0.3675%	0.3678%
	5.4259%	0.3680%	0.3666%	0.0014%	0.0009%	0.0024%
5 years	5.1657%	0.0543%	0.0555%	0.0545%	0.0555%	0.0543%
	5.6823%	0.0001%	0.0001%	0.4025%	0.4024%	0.4024%
	4.6491%	0.4022%	0.4023%	0.0001%	0.0000%	0.0000%
6 month	4.4942%	0.0181%	0.0190%	0.0192%	0.0190%	0.0181%
	4.9436%	0.0000%	0.0000%	0.4396%	0.4385%	0.4385%
	4.0448%	0.4373%	0.4385%	0.0000%	0.0000%	0.0000%
Age-group 60, female, volatility 2.11% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	0.8402%	0.0135%	0.0136%	0.0133%	0.0136%	0.0135%
	0.9243%	0.0012%	0.0012%	0.0520%	0.0522%	0.0522%
	0.7562%	0.0521%	0.0517%	0.0010%	0.0008%	0.0012%
5 years	0.6809%	0.0098%	0.0100%	0.0098%	0.0100%	0.0098%
	0.7490%	0.0002%	0.0002%	0.0533%	0.0532%	0.0532%
	0.6129%	0.0531%	0.0531%	0.0001%	0.0001%	0.0001%
6 month	0.5634%	0.0031%	0.0032%	0.0033%	0.0032%	0.0031%
	0.6198%	0.0000%	0.0000%	0.0552%	0.0550%	0.0550%
	0.5071%	0.0548%	0.0550%	0.0000%	0.0000%	0.0000%
Age-group 40, female, volatility 3.09% p.a. (flat)						

monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	0.0846% 0.0931% 0.0762%	0.0020% 0.0004% 0.0055%	0.0020% 0.0005% 0.0055%	0.0020% 0.0055% 0.0004%	0.0020% 0.0056% 0.0003%	0.0020% 0.0056% 0.0004%
5 years	0.0803% 0.0883% 0.0723%	0.0017% 0.0002% 0.0064%	0.0017% 0.0002% 0.0064%	0.0017% 0.0064% 0.0001%	0.0017% 0.0064% 0.0001%	0.0017% 0.0064% 0.0001%
6 month	0.0766% 0.0843% 0.0689%	0.0006% 0.0000% 0.0074%	0.0006% 0.0000% 0.0075%	0.0007% 0.0075% 0.0000%	0.0006% 0.0075% 0.0000%	0.0006% 0.0075% 0.0000%
Age-group 20, female, volatility 5.22% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	0.0236% 0.0260% 0.0213%	0.0009% 0.0004% 0.0018%	0.0009% 0.0004% 0.0018%	0.0009% 0.0018% 0.0004%	0.0009% 0.0019% 0.0004%	0.0009% 0.0018% 0.0004%
5 years	0.0225% 0.0248% 0.0203%	0.0008% 0.0002% 0.0019%	0.0008% 0.0003% 0.0019%	0.0008% 0.0020% 0.0002%	0.0008% 0.0020% 0.0002%	0.0008% 0.0020% 0.0002%
6 month	0.0215% 0.0237% 0.0194%	0.0003% 0.0000% 0.0021%	0.0003% 0.0000% 0.0021%	0.0003% 0.0021% 0.0000%	0.0003% 0.0021% 0.0000%	0.0003% 0.0021% 0.0000%
Age-group 0, female, volatility 2.80% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)						
10 years	0.4899% 0.5389% 0.4409%	0.0104% 0.0019% 0.0314%	0.0105% 0.0020% 0.0312%	0.0103% 0.0315% 0.0016%	0.0105% 0.0317% 0.0014%	0.0104% 0.0316% 0.0017%
5 years	0.4178% 0.4596% 0.3760%	0.0080% 0.0006% 0.0329%	0.0081% 0.0006% 0.0329%	0.0080% 0.0332% 0.0004%	0.0081% 0.0331% 0.0004%	0.0080% 0.0331% 0.0004%
6 month	0.3620% 0.3982% 0.3258%	0.0026% 0.0000% 0.0351%	0.0028% 0.0000% 0.0353%	0.0028% 0.0355% 0.0000%	0.0028% 0.0353% 0.0000%	0.0026% 0.0353% 0.0000%

Table 4. Scenario 1 option premiums, females

The options seem to be cheap at first glance. This is due to two facts; firstly, figure 12 shows that the volatility estimate appears to be too low, e.g. compared to the volatility of 5 year ATM EUR interest rate caplets (average = 19.82% p.a., max = 30.88% p.a., min = 10.74% p.a.; source Dresdner Kleinwort) and it is reasonable to expect increasing volatility when the market becomes liquid. However, as news usually causes volatility, the longevity market would probably be less volatile than interest rates. Secondly, for most age-groups, current mortality rates are very small and an increase by say 10% would be tiny in absolute terms and therefore low option prices are sensible.

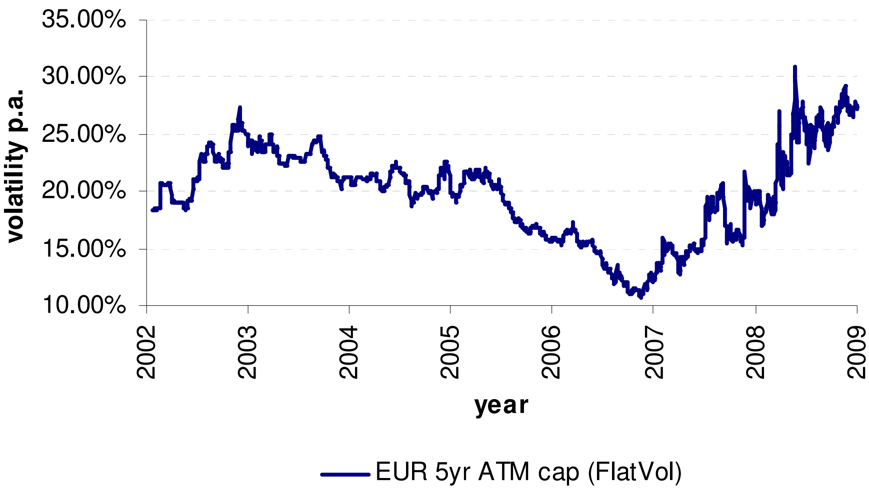

Figure 12. Historical implied volatilities 5yr ATM EUR caplets

Generally, the results are reasonable; options gain in value as time to maturity increases and relative option premiums increase as a function of volatility (e.g. *ATM call, females, age-group 80, 1.55% volatility p.a.*, relative premium $\frac{0.0713\%}{6.0288\%} = 1.18\%$ vs. *ATM call, females, age-group 20, 5.22% volatility p.a.*, relative premium = 3.81%). ATM calls and puts have similar values and ITM options have a higher premium over OTM options. Additionally, 10% ITM calls (puts) have approximately the same value as 10% ITM puts (calls). Put-call parity tends to hold within a 95% confidence level; only for 10 year OTM puts the premium as determined through put-call parity is slightly higher than their upper boundary. Finally, for 1,000 simulations, MCS strongly converge against B76 values. This is not surprising due to low volatility and therefore small standard errors. The MCS results will be less precise in more volatile markets but this can be handled by increasing the number of simulations.

SCENARIO 2

Applying the method described by Clewlow & Strickland (see Straja, 2001) the impact of 'shocks' can be modeled as follows:

$$p_{(t+1,x)} = p_{(t,x)} \times \exp\left[(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}\phi + \alpha(\psi_{(t,x)} - \lambda \times Y - \ln p_{(t,x)}) \times \delta t + Y \times q \right],$$

where α = mean reverting intensity (the higher α, the stronger mean reverting forces), λ = jump intensity (average number of jumps per path), Y = jump size and q = Poisson distributed random variables determined on the basis of λ and δt (i.e. 0 = no jump, 1 = jump)[7]. $\psi_{(t,x)}$ is the path the underlying will mean revert to. Clewlow & Strickland suggest to calculate $\psi_{(t,x)}$ as long term average of $\ln(p_{(t,x)})$. Since mortality rates follow a trend, long term averages appear to be inappropriate; alternatively, $\psi_{(t,x)}$ can be regarded as logarithm of periodic specific forecasted mortality rates (i.e. $\psi_{(t=n,x)} = \ln[p_{(t=0,x)} \times e^{(slope_x \times t_n)}]$) since it is reasonable to assume that the underlying will mean revert to this value (see Chapter 3, *unbiased expectation*).

[7] VBA codes for random Poisson variables see www.vbnumericalmethods.com/math/

Figure 13 (*males, age-group 80, 20% jump after 1 month*), shows that the mean reverting process $\alpha(\psi_{(t,x)} - \lambda \times Y - \ln p_{(t,x)}) \times \delta t$ tends to overshoot, especially for high α. This is an unwanted property as it illogical that jumps induce values below the trend thereafter and hence put values would be systematically too high.

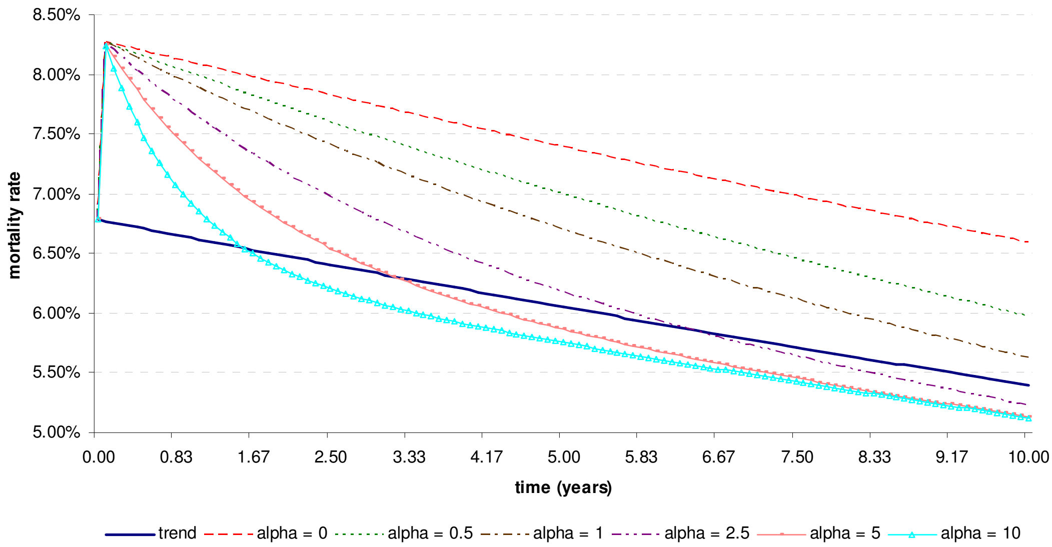

Figure 13. Mean reversion for different α

To alleviate this effect, the method suggested is adjusted by making mean reversion conditional to a previous jump (i.e. *if* jump in any previous period, *than* mean reversion, otherwise not). Nevertheless, the model still tends to overshoot for high α and thus one could either multiply $\psi_{(t,x)}$ with a (positive) mean reversion factor < 1 or (empirically) derive a more appropriate simulation input for $\psi_{(t,x)}$. Hence, accurate rather than intuitive estimates for $\psi_{(t,x)}$, λ, Y and α are potential areas for further research.

The option pricing is based on following simulation input: 'shock probability' = $\frac{1}{50}$ and thus $\lambda = \frac{1}{50} \times T$, Y = 20% (i.e. mortality rates instantaneously increase by 20% in case of a jump) and α = 0.50 (i.e. slow and almost linear mean reversion). For simplicity they are assumed to be identical for all age-groups and no distinction is made between males and females, although this would not hold in practice. For example, in case of a war, males are likely to be more affected than females; additionally, certain age-groups would probably suffer more. The option values are summarized in table 5 and 6.

Time to maturity	**Strike**	**Call (MCS)**	**Put (MCS)**
Age-group 80, male, volatility 1.60% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	8.9002%	0.1854%	0.0859%
	9.7902%	0.0289%	0.4692%
	8.0102%	0.6409%	0.0016%
5 years	7.7700%	0.1434%	0.0773%
	8.5470%	0.0185%	0.5575%
	6.9930%	0.6715%	0.0002%
6 month	6.8747%	0.0407%	0.0301%
	7.5622%	0.0049%	0.6653%
	6.1873%	0.6792%	0.0000%
Age-group 60, male, volatility 1.89% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	1.3723%	0.0311%	0.0158%
	1.5095%	0.0053%	0.0732%
	1.2351%	0.0992%	0.0007%
5 years	1.1930%	0.0242%	0.0140%
	1.3123%	0.0030%	0.0858%
	1.0737%	0.1032%	0.0001%
6 month	1.0515%	0.0067%	0.0055%
	1.1567%	0.0007%	0.1021%
	0.9464%	0.1040%	0.0000%
Age-group 40, male, volatility 3.74% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	0.1463%	0.0051%	0.0035%

	0.1609% 0.1317%	0.0017% 0.0114%	0.0090% 0.0009%
5 years	0.1419% 0.1560% 0.1277%	0.0045% 0.0010% 0.0127%	0.0033% 0.0109% 0.0004%
6 month	0.1380% 0.1518% 0.1242%	0.0016% 0.0001% 0.0136%	0.0014% 0.0134% 0.0000%
Age-group 20, male, volatility 4.26% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	0.0537% 0.0590% 0.0483%	0.0021% 0.0008% 0.0041%	0.0015% 0.0034% 0.0005%
5 years	0.0569% 0.0626% 0.0512%	0.0020% 0.0005% 0.0052%	0.0015% 0.0045% 0.0003%
6 month	0.0599% 0.0659% 0.0539%	0.0008% 0.0000% 0.0059%	0.0007% 0.0058% 0.0000%
Age-group 0, male, volatility 2.96% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	0.5903% 0.6493% 0.5313%	0.0175% 0.0046% 0.0442%	0.0109% 0.0338% 0.0018%
5 years	0.5117% 0.5629% 0.4605%	0.0138% 0.0023% 0.0448%	0.0094% 0.0378% 0.0006%
6 month	0.4499% 0.4949% 0.4049%	0.0039% 0.0002% 0.0443%	0.0037% 0.0438% 0.0000%

Table 5. Scenario 2 option premiums, males

Time to maturity	Strike	Call (MCS)	Put (MCS)
Age-group 80, female, volatility 1.55% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	6.0288% 6.6317% 5.4259%	0.1238% 0.0191% 0.4340%	0.0563% 0.3173% 0.0009%
5 years	5.1657% 5.6823% 4.6491%	0.0893% 0.0110% 0.4427%	0.0489% 0.3729% 0.0001%
6 month	4.4942% 4.9436% 4.0448%	0.0243% 0.0021% 0.4463%	0.0191% 0.4370% 0.0000%

Age-group 60, female, volatility 2.11% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	0.8402% 0.9243% 0.7562%	0.0203% 0.0038% 0.0611%	0.0108% 0.0453% 0.0007%
5 years	0.6809% 0.7490% 0.6129%	0.0130% 0.0013% 0.0568%	0.0093% 0.0506% 0.0001%
6 month	0.5634% 0.6198% 0.5071%	0.0037% 0.0006% 0.0558%	0.0032% 0.0544% 0.0000%
Age-group 40, female, volatility 3.09% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	0.0846% 0.0931% 0.0762%	0.0026% 0.0007% 0.0064%	0.0016% 0.0049% 0.0003%
5 years	0.0803% 0.0883% 0.0723%	0.0022% 0.0004% 0.0070%	0.0015% 0.0060% 0.0001%
6 month	0.0766% 0.0843% 0.0689%	0.0007% 0.0000% 0.0076%	0.0006% 0.0075% 0.0000%
Age-group 20, female, volatility 5.22% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	0.0236% 0.0260% 0.0213%	0.0011% 0.0005% 0.0020%	0.0008% 0.0017% 0.0003%
5 years	0.0225% 0.0248% 0.0203%	0.0009% 0.0003% 0.0021%	0.0008% 0.0019% 0.0002%
6 month	0.0215% 0.0237% 0.0194%	0.0004% 0.0000% 0.0021%	0.0003% 0.0021% 0.0000%
Age-group 0, female, volatility 2.80% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)			
10 years	0.4899% 0.5389% 0.4409%	0.0140% 0.0034% 0.0365%	0.0086% 0.0277% 0.0013%
5 years	0.4178% 0.4596% 0.3760%	0.0105% 0.0016% 0.0361%	0.0072% 0.0309% 0.0003%
6 month	0.3620% 0.3982% 0.3258%	0.0030% 0.0003% 0.0353%	0.0028% 0.0352% 0.0000%

Table 6. Scenario 2 option premiums, females

The results are sensible; as jumps are one-sided (i.e. only increasing mortality rates) call values gain on the expense of decreasing put values. However, due to increasing stochastic in conjunction with the asymmetric payoff of options, calls gain by more than puts lose in value. This also causes an increasing standard error and thus decreasing pricing accuracy. Notably, those one-sided mean reverting shocks would have a stronger positive impact on American, Bermudan and Asian calls as their payoff not solely depends on the final mortality rate, but also on previous levels.

SCENARIO 3

Scenario 3 extends the previous subsection to incorporate significant and instantaneous mortality improvements. This would cause a steeper slope of the trend and thus mortality rates would drop at a faster rate. This can either be modeled via slope dummies or non mean reverting jumps. The latter is chosen so that the process can be written as:

$$p_{(t+1,x)} = p_{(t,x)} \times \exp\left[(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}\phi + \alpha(\psi_{(t,x)} - \lambda \times Y - \ln p_{(t,x)}) \times \delta t + Y \times q - Y_2 \times q_2\right]$$

, where q_2 is determined on the basis of λ_2 (jump intensity of the second jump) and δt, and Y_2 = jump size of the second jump; the negative sign accounts for decreasing mortality rates in case of a jump. The non mean reversion condition means, that previous jumps are perfectly memorized. λ_2 is chosen to be $\frac{1}{25} \times T$ and Y_2 = 5.00%. The results are summarized in table 7 and 8.

Time to maturity	Strike	Call (MCS)	Put (MCS)	Put (PC parity)
Age-group 80, male, volatility 1.60% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	8.9002%	0.1230%	0.1538%	0.1528%
	9.7902%	0.0242%	0.5700%	0.5713%
	8.0102%	0.5362%	0.0072%	0.0071%
5 years	7.7700%	0.1329%	0.1148%	0.1149%
	8.5470%	0.0224%	0.6094%	0.6094%
	6.9930%	0.6241%	0.0009%	0.0008%
6 month	6.8747%	0.0403%	0.0374%	0.0375%
	7.5622%	0.0028%	0.6748%	0.6747%
	6.1873%	0.6739%	0.0000%	0.0000%
Age-group 60, male, volatility 1.89% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	1.3723%	0.0246%	0.0267%	0.0265%
	1.5095%	0.0046%	0.0911%	0.0910%
	1.2351%	0.0836%	0.0021%	0.0021%
5 years	1.1930%	0.0223%	0.0196%	0.0196%
	1.3123%	0.0037%	0.0938%	0.0940%
	1.0737%	0.0960%	0.0003%	0.0002%
6 month	1.0515%	0.0063%	0.0064%	0.0064%
	1.1567%	0.0001%	0.1036%	0.1035%
	0.9464%	0.1021%	0.0000%	0.0000%
Age-group 40, male, volatility 3.74% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	0.1463%	0.0042%	0.0045%	0.0044%
	0.1609%	0.0014%	0.0106%	0.0107%
	0.1317%	0.0099%	0.0013%	0.0014%
5 years	0.1419%	0.0042%	0.0039%	0.0040%
	0.1560%	0.0010%	0.0117%	0.0115%
	0.1277%	0.0119%	0.0005%	0.0004%
6 month	0.1380%	0.0016%	0.0015%	0.0015%
	0.1518%	0.0000%	0.0136%	0.0137%
	0.1242%	0.0134%	0.0000%	0.0000%
Age-group 20, male, volatility 4.26% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	0.0537%	0.0017%	0.0018%	0.0017%
	0.0590%	0.0007%	0.0040%	0.0041%
	0.0483%	0.0038%	0.0006%	0.0005%
5 years	0.0569%	0.0019%	0.0017%	0.0017%
	0.0626%	0.0005%	0.0048%	0.0049%
	0.0512%	0.0049%	0.0003%	0.0003%
6 month	0.0599%	0.0008%	0.0007%	0.0007%
	0.0659%	0.0000%	0.0059%	0.0059%
	0.0539%	0.0134%	0.0000%	0.0000%

Age-group 0, male, volatility 2.96% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	0.5903%	0.0139%	0.0150%	0.0153%
	0.6493%	0.0035%	0.0401%	0.0405%
	0.5313%	0.0378%	0.0033%	0.0031%
5 years	0.5117%	0.0127%	0.0115%	0.0115%
	0.5629%	0.0024%	0.0411%	0.0409%
	0.4605%	0.0419%	0.0009%	0.0008%
6 month	0.4499%	0.0041%	0.0039%	0.0039%
	0.4949%	0.0002%	0.0441%	0.0440%
	0.4049%	0.0134%	0.0000%	0.0000%

Table 7. Scenario 3 option premiums, males

Time to maturity	Strike	Call (MCS)	Put (MCS)	Put (PC parity)
Age-group 80, female, volatility 1.55% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	6.0288%	0.0960%	0.1028%	0.1028%
	6.6317%	0.0180%	0.3981%	0.3981%
	5.4259%	0.3663%	0.0051%	0.0051%
5 years	5.1657%	0.0899%	0.0748%	0.0750%
	5.6823%	0.0154%	0.4027%	0.4023%
	4.6491%	0.4183%	0.0009%	0.0007%
6 month	4.4942%	0.0227%	0.0243%	0.0242%
	4.9436%	0.0031%	0.4388%	0.4391%
	4.0448%	0.4371%	0.0000%	0.0000%
Age-group 60, female, volatility 2.11% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	0.8402%	0.0155%	0.0166%	0.0168%
	0.9243%	0.0033%	0.0552%	0.0555%
	0.7562%	0.0507%	0.0018%	0.0016%
5 years	0.6809%	0.0140%	0.0120%	0.0118%
	0.7490%	0.0024%	0.0534%	0.0536%
	0.6129%	0.0554%	0.0004%	0.0004%
6 month	0.5634%	0.0035%	0.0036%	0.0035%
	0.6198%	0.0003%	0.0554%	0.0555%
	0.5071%	0.0549%	0.0000%	0.0000%
Age-group 40, female, volatility 3.09% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	0.0846%	0.0021%	0.0022%	0.0021%
	0.0931%	0.0006%	0.0057%	0.0058%
	0.0762%	0.0056%	0.0005%	0.0005%
5 years	0.0803%	0.0021%	0.0019%	0.0020%
	0.0883%	0.0004%	0.0065%	0.0066%
	0.0723%	0.0067%	0.0002%	0.0002%

6 month	0.0766% 0.0843% 0.0689%	0.0007% 0.0000% 0.0075%	0.0007% 0.0075% 0.0000%	0.0007% 0.0073% 0.0000%
Age-group 20, female, volatility 5.22% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	0.0236% 0.0260% 0.0213%	0.0009% 0.0004% 0.0018%	0.0010% 0.0019% 0.0004%	0.0010% 0.0018% 0.0005%
5 years	0.0225% 0.0248% 0.0203%	0.0009% 0.0003% 0.0020%	0.0008% 0.0020% 0.0002%	0.0009% 0.0018% 0.0002%
6 month	0.0215% 0.0237% 0.0194%	0.0003% 0.0000% 0.0021%	0.0003% 0.0021% 0.0000%	0.0003% 0.0020% 0.0000%
Age-group 0, female, volatility 2.80% p.a. (flat) monthly time steps (10 years and 5 years) / daily time steps (6 month)				
10 years	0.4899% 0.5389% 0.4409%	0.0111% 0.0028% 0.0315%	0.0119% 0.0331% 0.0021%	0.0117% 0.0328% 0.0020%
5 years	0.4178% 0.4596% 0.3760%	0.0102% 0.0020% 0.0344%	0.0090% 0.0333% 0.0006%	0.0092% 0.0333% 0.0005%
6 month	0.3620% 0.3982% 0.3258%	0.0030% 0.0003% 0.0352%	0.0029% 0.0354% 0.0000%	0.0029% 0.0356% 0.0000%

Table 8. Scenario 3 option premiums, females

Comparing the results to those of scenario 2, they appear to be sensible. Calls lose in value relative to puts as jumps are now two-sided. Notably, when setting $\lambda_1 = \lambda_2$, $Y_1 = Y_2$ and $\alpha = 0$, ATM puts ≈ ATM calls but the values are higher than those in scenario 1. This is due to the fact that the incorporation of random Poisson variables increases the standard deviation of option values, leading to increasing premiums. Again, put-call parity holds; however, compared to scenario 1, the forward is determined as average of final mortality rates $p_{(T,x)}$ across 1,000 simulations at $\sigma = 0.00\%$ (i.e. random Poisson variables are the only remaining stochastic).

To sum up, the proposed option pricing framework appears to be very flexible. It can be easily calibrated for individual thoughts regarding the trend, the impact of 'shocks', expectations about future mortality improvements and biological reasonableness.

SENSITIVITY ANALYSIS

Knowing the sensitivity of the option prices to changes in input parameters is important for two reasons; firstly, the greeks are important for traders to manage the risk of their books. Secondly, the analysis of sensitivity to other input parameters identifies which estimates are important and which can be neglected and thus allows prioritizing areas of further research. Estimates for MCS sensitivities are obtained by applying the technique of numerical differentiation.

THE GREEKS

The greeks show the change in the value of an option with respect to changes in the underlying (*Delta*, Δ), time (*Theta*, Θ), Delta (*Gamma*, Γ), volatility (*Vega*, ν) and interest rates (*Rho*, ρ) (Hull, 2009). In a MCS they are estimated by performing the following 6 steps (i.e. $\frac{\hat{f}^* - \hat{f}}{\Delta x}$, where $\hat{f}^*$ = new value for the derivative, $\hat{f}$ = base case estimate and Δx = small change in the value of an underlying variable):

1. Calculate base case option value

2. Store the random numbers and Poisson variables from 1.
3. Successively shift input parameters (spot price for *Delta* and *Gamma*, volatility for *Vega*, interest rates for *Rho* and time to maturity for *Theta*) by a small amount (e.g. 1%), keeping everything else equal
4. Calculate the option prices for each scenario
5. Calculate the difference between the option priced under 4. and the base case option price
6. The sensitivity to changes in the input parameters is obtained via dividing the option price differential from 5. by Δx

The base case option value is calculated for *males, age-group 80,* with following input: $T = 60\,month$, $p_{(t=0,80)} = 6.78\%$, $slope = -2.28\%\ p.a.$, $r = 5.00\%\ p.a.$ and thus $drift = 2.72\%\ p.a.$, $K = 7.75\%$, $\sigma = 15.00\%\ p.a.$, mean reverting jump: $\lambda_1 = 0.10$, $Y_1 = 20.00\%$ and $\alpha = 0.50$, non mean reverting jump: $\lambda_2 = 0.20$ and $Y_2 = 5.00\%$. The sensitivity analysis is performed on the basis of 5,000 simulations. The respective option premiums and their boundaries for a 95% confidence level are 0.8162% (0.7326% - 0.8999%) for the call and 0.7676% (0.7097% - 0.8255%) for the put. The results for the greeks are exhibited in figures 14 – 18 (details see Appendix F).

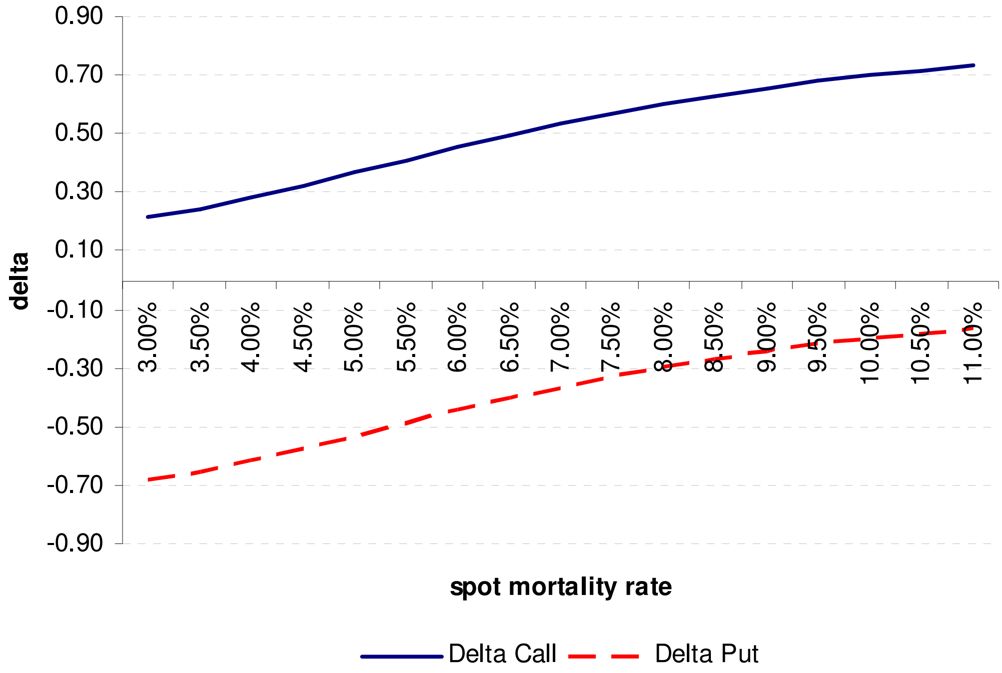

Figure 14. Delta (per £)

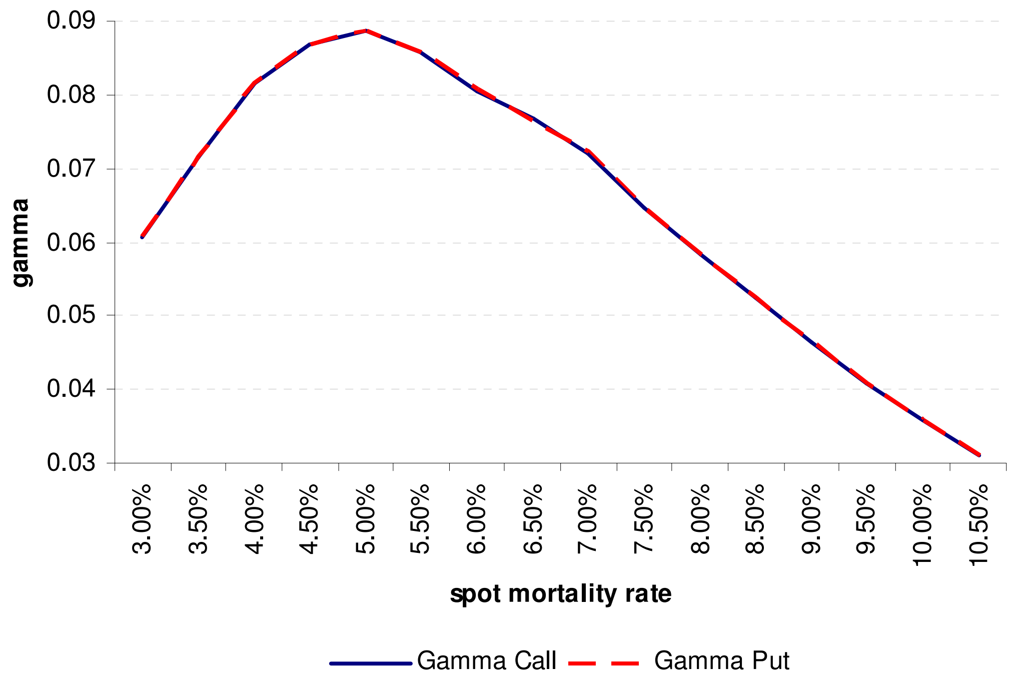

Figure 15. Gamma (£ per £)

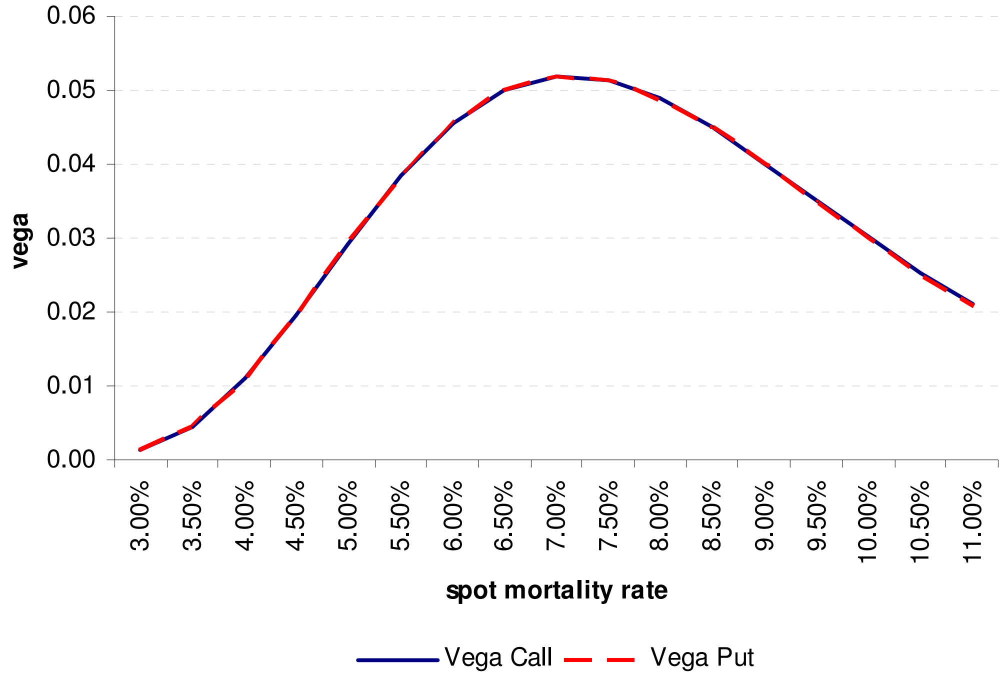

Figure 16. Vega (per %)

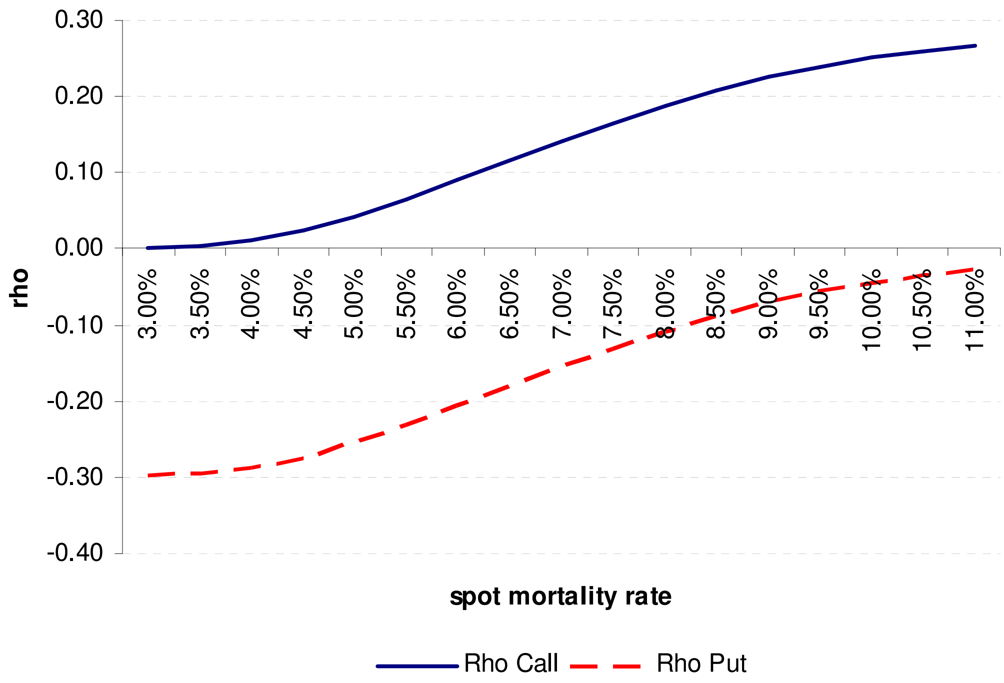

Figure 17. Rho (per %)

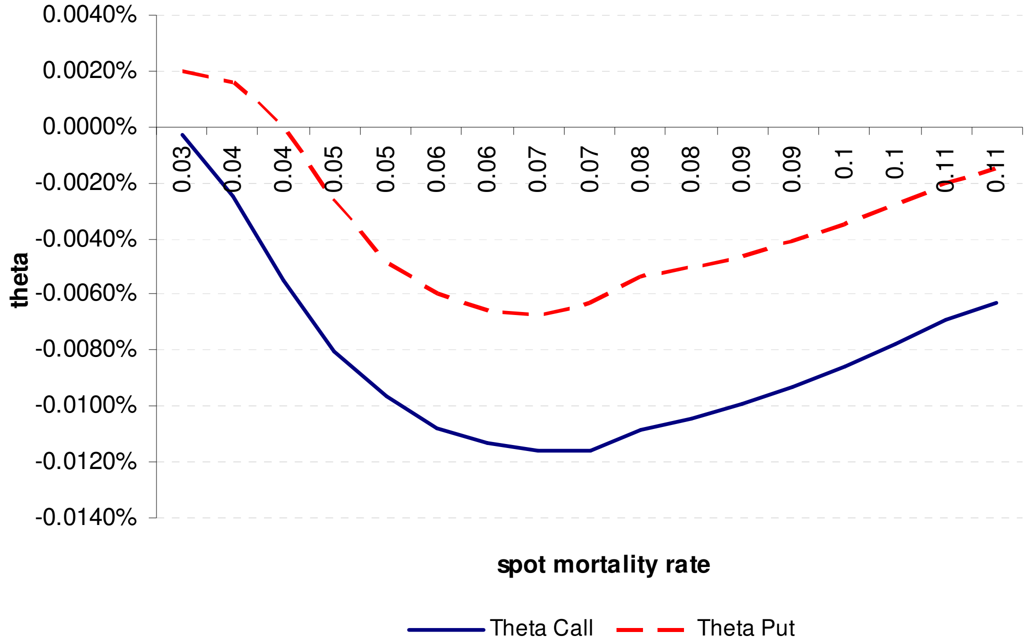

Figure 18. Theta (per month)

The results are sensible: *Delta* is ≈ 0.50 for ATM options, converges against 1 (-1) for ITM calls (puts) and against 0 for OTM options. *Vegas* are positive for puts and calls as both gain in value when volatility increases due to the asymmetric payoff of options and peak ATM. *Rho* is positive (negative) for calls (puts) as they gain (lose) in value when interest rates increase. It is also larger for ITM options (due to cost of carry; ITM options require more cash) and decreases steadily as the option moves OTM. *Theta* approaches 0 for OTM options, meaning that they are insensitive to changes in time. The loss in time value is highest for ATM options an *Theta* can be positive for ITM puts due to limited upside (mortality rates have a natural boundary of 0.00%). Is also tends to be negative when *Gamma* is positive (and vice versa).

Nevertheless, compared against sensitivities from closed-form solutions which are mathematically obtained through partial derivatives, simulation results tend to emphasize the error in the MCS value; the MCS *Gamma* is least accurate as the error is amplified (*Gamma* = 1st derivative of already erroneous *Delta*). For example, the peak in figure 15 is too far on the left as *Gamma* should be highest for ATM options. The standard error for 5,000 simulations is 0.0191% (call) and 0.0132% (put) respectively, indicating the uncertainty of the sample value compared to the true value. Since the greeks (especially *Delta*, *Vega* and *Gamma*) are crucial from a hedging perspective, imprecise MCS values make it necessary to simulate with a large number of sample paths. Practitioners regard this as a key problem with using MCS methods as this approach is quite time-consuming and thus not applicable in fast changing markets. Thus, closed-form solutions should be preferred over MCS where available.

SENSITIVITY TO OTHER INPUT PARAMETERS

Sensitivities are derived by applying the same method and simulation input as in the previous subsection (i.e. base case call (put) value = 0.8162% (0.7676%)). Consecutively, λ_1, Y_1, α, $\psi_{(t,x)}$, λ_2 and Y_2 are shifted by +/- 5% keeping everything else equal. The results are summarized in table 9.

Parameter	Call value	Sensitivity	Put value	Sensitivity
@ λ = 0.1050 **@ λ = 0.0950**	0.8199% 0.8154%	0.0074 0.0016	0.7636% 0.7677%	-0.0080 -0.0002
@ Y = 21.00% **@ Y =**	0.8197% 0.8128%	0.0034 0.0034	0.7661% 0.7691%	-0.0015 -0.0015

19.00%				
@ α = 0.5250 **@ α = 0.4750**	0.8143% 0.8182%	-0.0008 -0.0008	0.7678% 0.7673%	0.0001 0.0001
@ ψ = -2.6994 **@ ψ = -2.7995**	0.8205% 0.8119%	0.0009 0.0008	0.7654% 0.7700%	-0.0004 -0.0005
@ λ_2 = 0.2100 **@ λ_2 = 0.1900**	0.8152% 0.8164%	-0.0010 -0.0002	0.7676% 0.7650%	0.0014 0.0026
@ Y_2 = 5.25% **@ Y_2 = 4.75%**	0.8148% 0.8176%	-0.0056 -0.0057	0.7691% 0.7661%	0.0059 0.0058

Table 9. Sensitivity to other input parameters

The results appear to be logical; call (put) values gain (lose) in value when either λ_1 or Y_1 increases. This is due to the fact that this jump (increasing mortality rates) by itself is one-sided. When α is shifted upwards, mean reverting forces become stronger and hence call (put) premiums go down (up). An increasing ψ means that, after a shock, the underlying will mean revert to a higher level and thus promotes the value of calls whereas puts decrease in value. The analysis of the second jump delivers reverse results; when λ_2 or Y_2 increase, puts become more expensive on the expense of decreasing call prices. This is sensible as the jump himself is one-sided as well (decreasing mortality rates). Notably, the sensitivities to lambdas are exposed to changing randomness in Poisson variables, diluting the findings. The validity can be improved by running a higher number of simulations.

The analysis shows, that accurate estimates for λ_1, Y_1, λ_2 and Y_2 are more important than those for α and $\psi_{(t,x)}$. However, bearing the overshooting (see '*Scenario 2*') in mind, this might not be true for high α. To improve the robustness, further work should therefore focus on this issue fist.

CHAPTER 5:

CONCLUSION

This paper has outlined the need for longevity derivatives, with an emphasis on longevity options. It has been shown that they can be valued arbitrage-free under the prerequisite of liquid longevity futures. Afterwards a universal applicable mixed jump-diffusion model has been introduced; the advantage of the approach suggested is that it can easily be calibrated for country- and case-specific data as well as for individual expectations regarding future mortality. Additionally, the use of mortality instead of survival rates is beneficial as it is very intuitive from a capital markets perspective due to similarities with the credit world. Those similarities can be made use of to transfer applications from the credit to the longevity market (e.g. n^{th}-to-default baskets). Structured products are important as they allow creating specific risk-return characteristics and thus would ease the transfer of longevity risk to investors. This paper has also outlined fields for potential future research, such as volatility and simulation input parameter estimates. Another issue for further work is the overshooting for high a. This work does not claim to be complete, but it is a good starting point for further developments in longevity options.

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APPENDIX A:

BASIC DEFINITIONS

The **age-group approach** tracks mortality rates for a constant age-group over time (e.g. 30 year old male). **Cohort:** A group of lives categorized according to common characteristics such as sex and year of birth. The *cohort approach* tracks mortality rates for a given year of birth over time (e.g. males born in 1979). **Life annuity:** In its basic form, the insured makes (single or a series of regular) payments into the insurance and, in return, the insurer is obliged to pay annuities beginning from a pre-defined future date until the death of the insured. **Life expectancy (LE):** The average remaining lifetime for an individual expressed in years. *Period life expectancy* uses the current mortality table without further improvements whereas *cohort life expectancy* incorporates expected future mortality improvements. **Life insurance:** provides coverage for a pre-defined period of time. If the insured dies during this period, the insurer pays death benefits the beneficiary. Notably, various types of life insurances / annuities exist, but for the purpose of this paper those basic definitions are sufficient. **Longevity risk** (where longevity refers to the length of life) is typically borne by annuity providers or pension funds from an unanticipated reduction in mortality rates. That is, a loss is sustained if longevity increases (i.e. mortality rate falls). **Mortality improvement:** Rate of decrease in mortality rate, usually in respect of the progression of time. **Mortality probability/rate** (also *initial rate of mortality* or *central rate of mortality*): The proportion of people currently alive, for a specific age, population, gender, as published for a given index reference year, that are expected to die within the year. The initial rate of mortality is normally denoted as q_x for lives aged x. **Mortality risk:** The risk typically borne by life insurance providers from an unanticipated increase in mortality rates. That is, a loss is sustained if mortality increases. **Survival probability/rate:** The annual survival probability for an individual. The survival probability p_x at age x is $1 - q_x$ (JP Morgan, 2009).

APPENDIX B:

BACKTESTING SUMMARY

Backtest (male)	Linear (average error)	Logarithmic (average error)	Exponential (average error)
1967	0.1911%	0.1148%	0.1810%
1968	0.1520%	0.1290%	0.1506%
1969	0.1892%	0.1503%	0.1905%
1970	0.2480%	0.1963%	0.2549%
1971	0.2666%	0.2255%	0.2738%
1972	0.2836%	0.2604%	0.2893%
1973	0.2755%	0.2849%	0.2798%
1974	0.2742%	0.3148%	0.2780%
1975	0.2312%	0.3114%	0.2351%
1976	0.1852%	0.2886%	0.1871%
1977	0.1529%	0.2620%	0.1525%
1978	0.1374%	0.2394%	0.1338%
1979	0.1381%	0.2414%	0.1324%
1980	0.1362%	0.2401%	0.1294%
1981	0.1334%	0.2390%	0.1267%
1982	0.1280%	0.2462%	0.1246%

1983	0.1193%	0.2424%	0.1159%
1984	0.1169%	0.2494%	0.1123%
1985	0.1303%	0.2626%	0.1265%
1986	0.1517%	0.2781%	0.1459%
1987	0.1351%	0.2716%	0.1273%
1988	0.1110%	0.2492%	0.0984%
1989	0.1022%	0.2368%	0.0856%
1990	0.1020%	0.2345%	0.0858%
1991	0.0934%	0.2292%	0.0777%
1992	0.0853%	0.2338%	0.0749%
1993	0.0793%	0.2327%	0.0692%
1994	0.0765%	0.2474%	0.0782%
1995	0.0857%	0.2677%	0.0979%
1996	0.1023%	0.2908%	0.1177%
Average	**0.1538%**	**0.2423%**	**0.1511%**
Min	**0.0765% (year 1994)**	**0.1148% (year 1967)**	**0.0692% (year 1993)**
Max	**0.2836%**	**0.3148%**	**0.2893%**

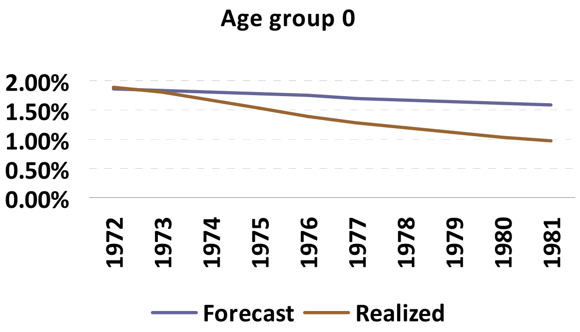
Backtest (female)	**Linear (average error)**	**Logarithmic (average error)**	**Exponential (average error)**
1967	0.1922%	0.0851%	0.1587%
1968	0.1149%	0.1162%	0.1001%
1969	0.0930%	0.1454%	0.0924%
1970	0.1512%	0.2039%	0.1559%

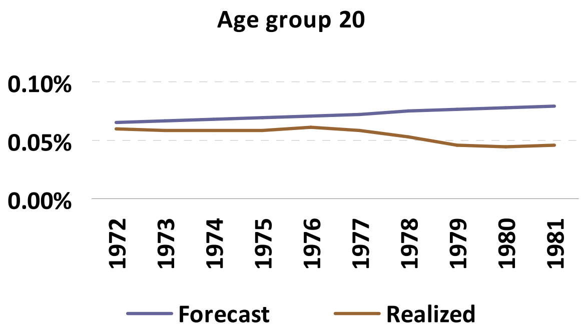
1971	0.2165%	0.2581%	0.2199%
1972	0.2418%	0.2951%	0.2438%
1973	0.2311%	0.3138%	0.2337%
1974	0.2228%	0.3349%	0.2273%
1975	0.1863%	0.3321%	0.1960%
1976	0.1272%	0.3083%	0.1442%
1977	0.0724%	0.2782%	0.0944%
1978	0.0385%	0.2451%	0.0509%
1979	0.0418%	0.2336%	0.0409%
1980	0.0478%	0.2240%	0.0349%
1981	0.0534%	0.2159%	0.0331%
1982	0.0558%	0.2083%	0.0336%
1983	0.0678%	0.1892%	0.0335%
1984	0.0698%	0.1781%	0.0357%
1985	0.0656%	0.1769%	0.0364%
1986	0.0665%	0.1735%	0.0365%
1987	0.0806%	0.1588%	0.0424%
1988	0.1113%	0.1334%	0.0651%
1989	0.1174%	0.1213%	0.0703%
1990	0.1003%	0.1210%	0.0571%
1991	0.0799%	0.1238%	0.0437%
1992	0.0580%	0.1316%	0.0355%
1993	0.0500%	0.1346%	0.0377%

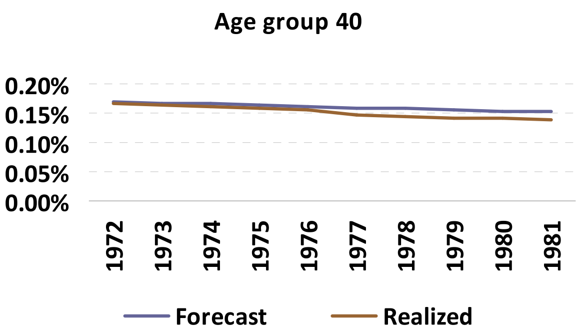
1994 1995 1996	0.0471% 0.0479% 0.0502%	0.1428% 0.1542% 0.1632%	0.0416% 0.0491% 0.0539%
Average	**0.1033%**	**0.1967%**	**0.0899%**
Min	**0.0385%**	**0.0851%**	**0.0331%**
Max	**0.2418%**	**0.3349%**	**0.2438%**

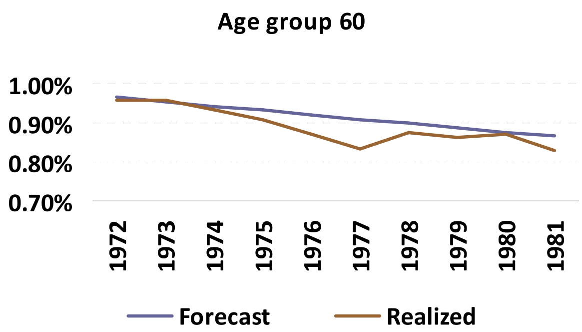
Appendix C1:

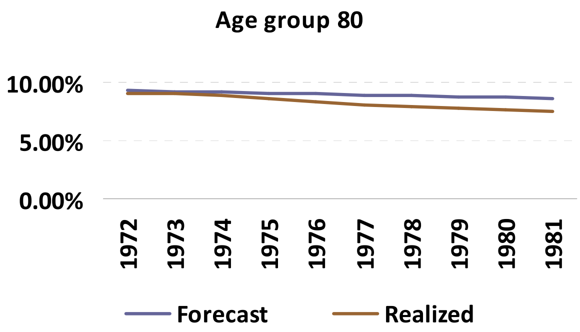
Worst Forecast, Female (Backtest 1972)





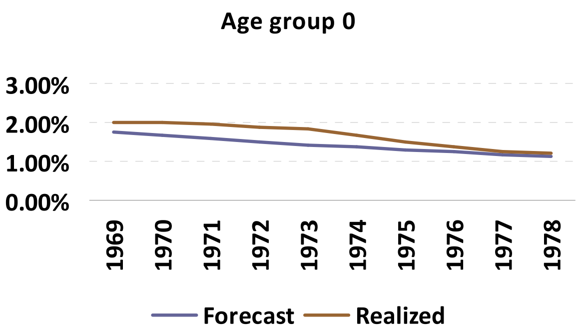


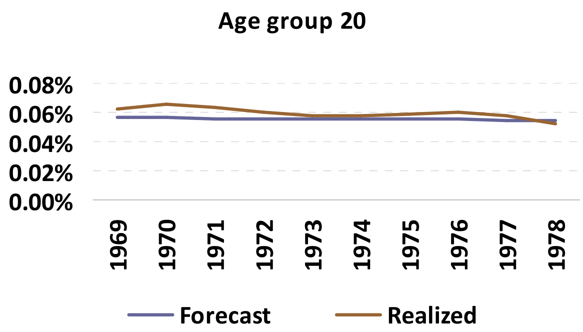


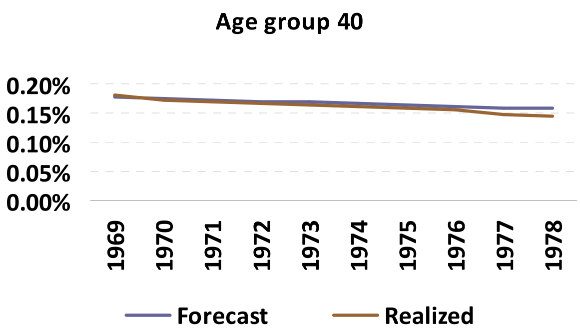


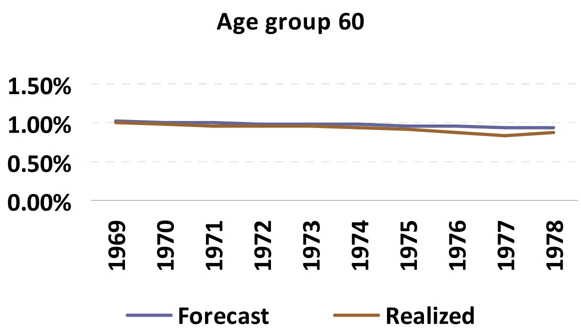
Appendix C2:

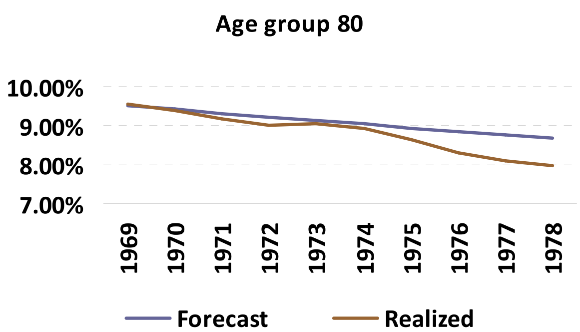
Average Forecast, Female (Backtest 1969)





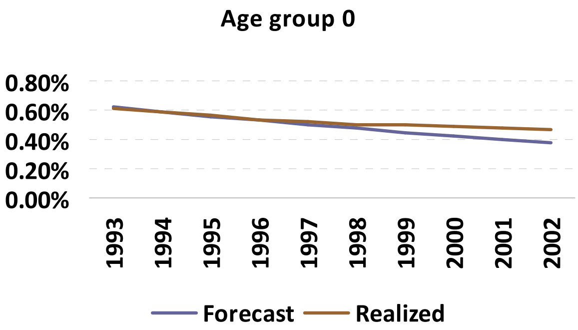


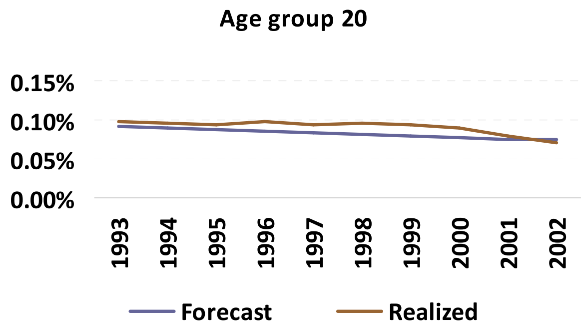


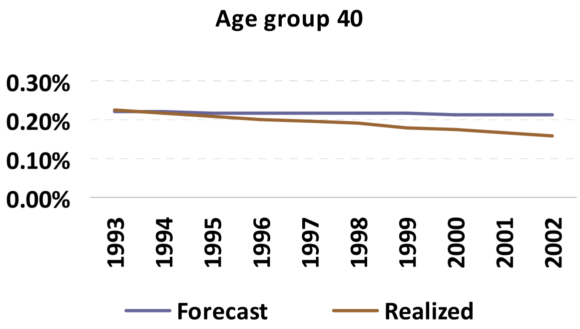


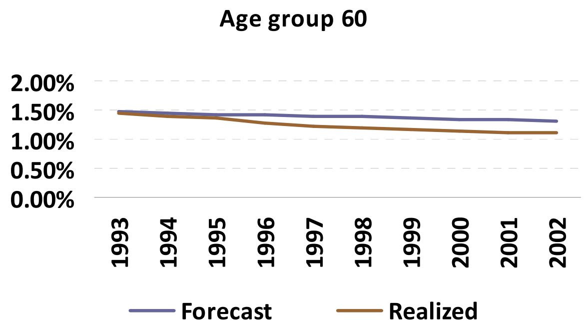
APPENDIX C3:

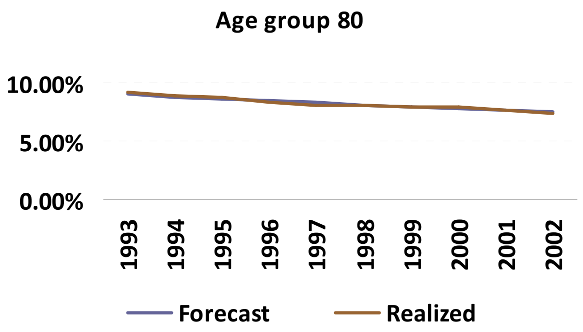
BEST FORECAST, MALE (BACKTEST 1993)





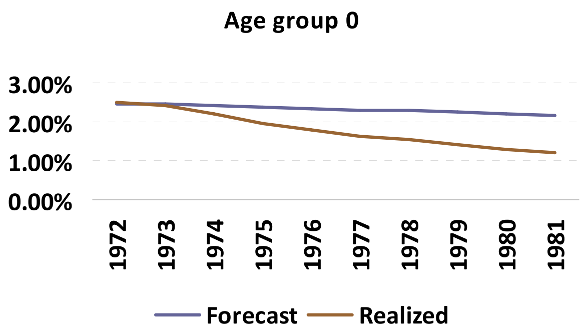


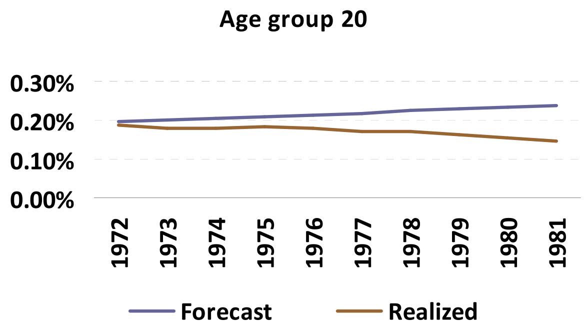


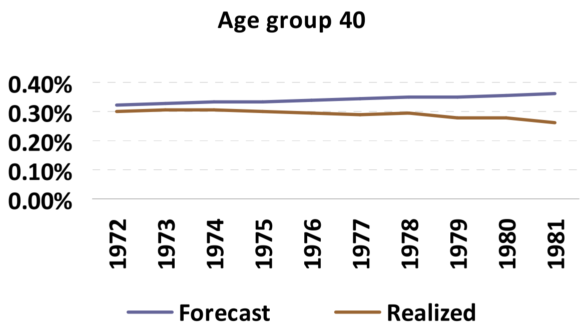


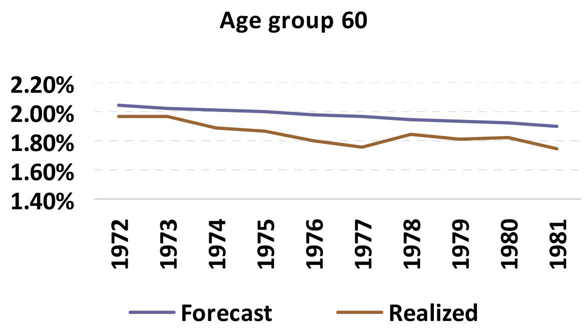
APPENDIX C4:

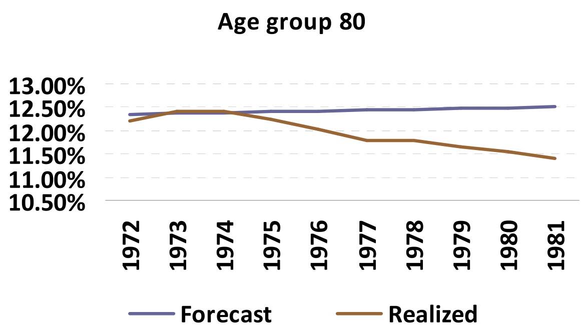
WORST FORECAST, MALE (BACKTEST 1993)





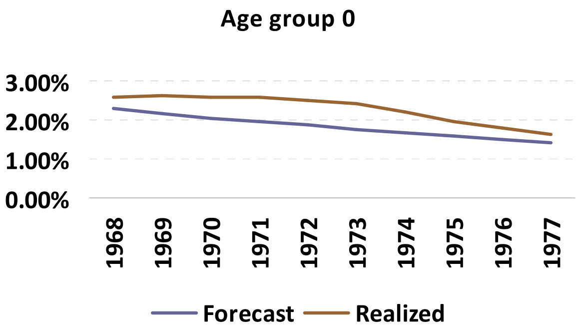


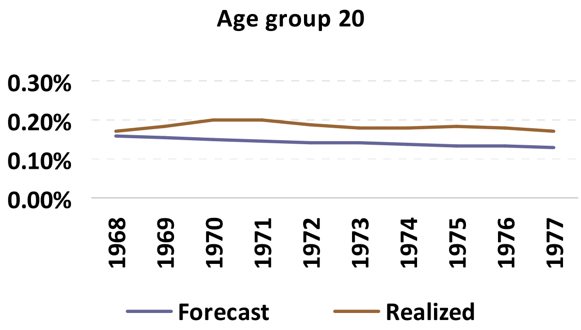


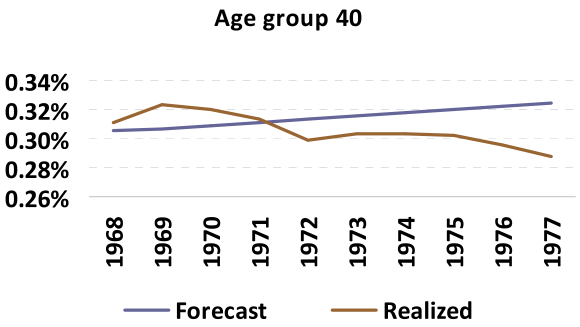


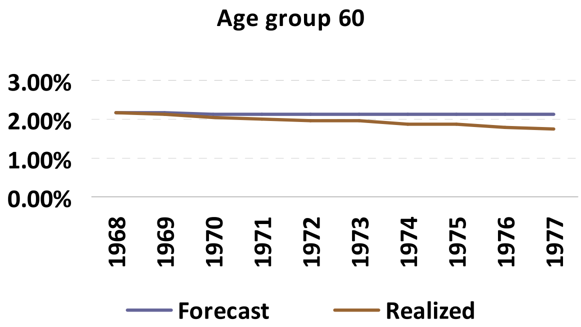
Appendix C5:

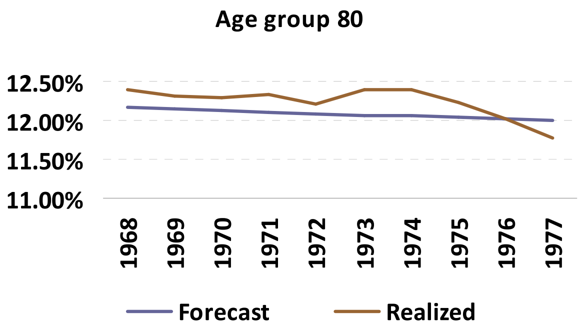
Average Forecast, Male (Backtest 1968)











APPENDIX D1:

WEIBULL DISTRIBUTION 2005, MALE

Estimates based on intervals of five age-groups each ([0;4], …, [85;89]). E.g., slope b and intercept a are identical for $\hat{F}(0)$ and $\hat{F}(4)$.

t	b	a	$\hat{F}(t)$	F(t)	$\hat{F}(t)$ - F(t)
0	-0.5705	-8.0350	0.4473%	0.4436%	0.0037%
1	-0.5705	-8.0350	0.0324%	0.0373%	-0.0049%
2	-0.5705	-8.0350	0.0218%	0.0175%	0.0043%
3	-0.5705	-8.0350	0.0173%	0.0177%	-0.0004%
4	-0.5705	-8.0350	0.0147%	0.0157%	-0.0010%
5	-0.3686	-8.3782	0.0127%	0.0135%	-0.0008%
6	-0.3686	-8.3782	0.0119%	0.0108%	0.0010%
7	-0.3686	-8.3782	0.0112%	0.0108%	0.0004%
8	-0.3686	-8.3782	0.0107%	0.0115%	-0.0009%
9	-0.3686	-8.3782	0.0102%	0.0101%	0.0001%
10	1.6448	-13.1618	0.0085%	0.0084%	0.0001%
11	1.6448	-13.1618	0.0099%	0.0094%	0.0005%
12	1.6448	-13.1618	0.0115%	0.0129%	-0.0015%
13	1.6448	-13.1618	0.0131%	0.0126%	0.0005%

14	1.6448	-13.1618	0.0148%	0.0145%	0.0003%
15	4.9465	-21.9007	0.0202%	0.0193%	0.0009%
16	4.9465	-21.9007	0.0278%	0.0308%	-0.0029%
17	4.9465	-21.9007	0.0376%	0.0327%	0.0048%
18	4.9465	-21.9007	0.0499%	0.0593%	-0.0094%
19	4.9465	-21.9007	0.0652%	0.0597%	0.0055%
20	-0.1724	-6.8974	0.0603%	0.0603%	0.0000%
21	-0.1724	-6.8974	0.0598%	0.0596%	0.0001%
22	-0.1724	-6.8974	0.0593%	0.0605%	-0.0012%
23	-0.1724	-6.8974	0.0588%	0.0568%	0.0021%
24	-0.1724	-6.8974	0.0584%	0.0594%	-0.0010%
25	0.1543	-7.8974	0.0610%	0.0597%	0.0013%
26	0.1543	-7.8974	0.0614%	0.0636%	-0.0021%
27	0.1543	-7.8974	0.0618%	0.0609%	0.0009%
28	0.1543	-7.8974	0.0621%	0.0629%	-0.0008%
29	0.1543	-7.8974	0.0625%	0.0617%	0.0007%
30	1.3465	-11.8855	0.0671%	0.0659%	0.0013%
31	1.3465	-11.8855	0.0702%	0.0734%	-0.0032%
32	1.3465	-11.8855	0.0732%	0.0712%	0.0021%
33	1.3465	-11.8855	0.0763%	0.0763%	0.0000%
34	1.3465	-11.8855	0.0795%	0.0797%	-0.0003%
35	3.2895	-18.7858	0.0833%	0.0844%	-0.0012%
36	3.2895	-18.7858	0.0913%	0.0909%	0.0005%

37	3.2895	-18.7858	0.0999%	0.0987%	0.0012%
38	3.2895	-18.7858	0.1091%	0.1073%	0.0019%
39	3.2895	-18.7858	0.1188%	0.1213%	-0.0025%
40	4.9490	-24.8691	0.1342%	0.1375%	-0.0033%
41	4.9490	-24.8691	0.1517%	0.1477%	0.0040%
42	4.9490	-24.8691	0.1708%	0.1684%	0.0025%
43	4.9490	-24.8691	0.1919%	0.1939%	-0.0019%
44	4.9490	-24.8691	0.2150%	0.2165%	-0.0015%
45	5.7977	-28.0981	0.2407%	0.2389%	0.0018%
46	5.7977	-28.0981	0.2734%	0.2726%	0.0008%
47	5.7977	-28.0981	0.3096%	0.3140%	-0.0044%
48	5.7977	-28.0981	0.3498%	0.3537%	-0.0039%
49	5.7977	-28.0981	0.3941%	0.3883%	0.0058%
50	5.0037	-25.0346	0.4244%	0.4206%	0.0038%
51	5.0037	-25.0346	0.4685%	0.4746%	-0.0061%
52	5.0037	-25.0346	0.5162%	0.5206%	-0.0045%
53	5.0037	-25.0346	0.5677%	0.5566%	0.0110%
54	5.0037	-25.0346	0.6231%	0.6276%	-0.0045%
55	4.7388	-23.9862	0.6742%	0.6813%	-0.0071%
56	4.7388	-23.9862	0.7341%	0.7375%	-0.0035%
57	4.7388	-23.9862	0.7981%	0.7704%	0.0276%
58	4.7388	-23.9862	0.8663%	0.8783%	-0.0120%
59	4.7388	-23.9862	0.9391%	0.9450%	-0.0060%

60	5.0730	-25.3332	1.0379%	1.0371%	0.0008%
61	5.0730	-25.3332	1.1282%	1.1180%	0.0102%
62	5.0730	-25.3332	1.2246%	1.2386%	-0.0140%
63	5.0730	-25.3332	1.3275%	1.3383%	-0.0108%
64	5.0730	-25.3332	1.4371%	1.4234%	0.0137%
65	5.5873	-27.4642	1.5784%	1.5899%	-0.0114%
66	5.5873	-27.4642	1.7178%	1.7099%	0.0078%
67	5.5873	-27.4642	1.8669%	1.8498%	0.0171%
68	5.5873	-27.4642	2.0264%	2.0325%	-0.0061%
69	5.5873	-27.4642	2.1967%	2.2045%	-0.0078%
70	7.3702	-34.9990	2.4739%	2.4709%	0.0029%
71	7.3702	-34.9990	2.7427%	2.7594%	-0.0167%
72	7.3702	-34.9990	3.0360%	3.0364%	-0.0004%
73	7.3702	-34.9990	3.3553%	3.3091%	0.0462%
74	7.3702	-34.9990	3.7026%	3.7354%	-0.0329%
75	7.3962	-35.0808	4.2047%	4.1902%	0.0145%
76	7.3962	-35.0808	4.6273%	4.6234%	0.0039%
77	7.3962	-35.0808	5.0849%	5.1090%	-0.0242%
78	7.3962	-35.0808	5.5796%	5.6191%	-0.0395%
79	7.3962	-35.0808	6.1137%	6.0681%	0.0456%
80	9.1551	-42.7774	6.7583%	6.7833%	-0.0249%
81	9.1551	-42.7774	7.5409%	7.5319%	0.0090%
82	9.1551	-42.7774	8.3987%	8.3626%	0.0361%

83	9.1551	-42.7774	9.3370%	9.3128%	0.0242%
84	9.1551	-42.7774	10.3610%	10.4066%	-0.0456%
85	7.3328	-34.6699	11.6026%	11.6235%	-0.0209%
86	7.3328	-34.6699	12.5735%	12.5873%	-0.0138%
87	7.3328	-34.6699	13.6067%	13.6388%	-0.0321%
88	7.3328	-34.6699	14.7042%	14.4826%	0.2216%
89	7.3328	-34.6699	15.8681%	16.0262%	-0.1581%

APPENDIX D2:

WEIBULL DISTRIBUTION 2005, FEMALE

t	b	a	$\hat{F}(t)$	F(t)	$\hat{F}(t)$ - F(t)
0	-0.5761	-8.2571	0.3676%	0.3563%	0.0113%
1	-0.5761	-8.2571	0.0259%	0.0293%	-0.0034%
2	-0.5761	-8.2571	0.0174%	0.0180%	-0.0006%
3	-0.5761	-8.2571	0.0138%	0.0135%	0.0003%
4	-0.5761	-8.2571	0.0117%	0.0105%	0.0011%
5	-0.6324	-8.2181	0.0097%	0.0092%	0.0005%
6	-0.6324	-8.2181	0.0087%	0.0095%	-0.0008%
7	-0.6324	-8.2181	0.0079%	0.0080%	-0.0001%
8	-0.6324	-8.2181	0.0072%	0.0067%	0.0005%
9	-0.6324	-8.2181	0.0067%	0.0069%	-0.0001%
10	1.3090	-12.5145	0.0075%	0.0077%	-0.0002%
11	1.3090	-12.5145	0.0085%	0.0090%	-0.0005%
12	1.3090	-12.5145	0.0095%	0.0081%	0.0014%
13	1.3090	-12.5145	0.0105%	0.0108%	-0.0002%
14	1.3090	-12.5145	0.0116%	0.0123%	-0.0006%
15	2.6215	-16.0312	0.0132%	0.0141%	-0.0009%

16	2.6215	-16.0312	0.0156%	0.0139%	0.0018%
17	2.6215	-16.0312	0.0183%	0.0183%	0.0000%
18	2.6215	-16.0312	0.0213%	0.0239%	-0.0026%
19	2.6215	-16.0312	0.0245%	0.0233%	0.0013%
20	0.3311	-9.4475	0.0213%	0.0214%	-0.0002%
21	0.3311	-9.4475	0.0216%	0.0206%	0.0010%
22	0.3311	-9.4475	0.0220%	0.0229%	-0.0010%
23	0.3311	-9.4475	0.0223%	0.0228%	-0.0006%
24	0.3311	-9.4475	0.0226%	0.0219%	0.0006%
25	1.7095	-13.9657	0.0211%	0.0215%	-0.0004%
26	1.7095	-13.9657	0.0226%	0.0232%	-0.0007%
27	1.7095	-13.9657	0.0241%	0.0232%	0.0009%
28	1.7095	-13.9657	0.0256%	0.0231%	0.0025%
29	1.7095	-13.9657	0.0272%	0.0298%	-0.0025%
30	1.8990	-14.5337	0.0311%	0.0324%	-0.0013%
31	1.8990	-14.5337	0.0331%	0.0310%	0.0021%
32	1.8990	-14.5337	0.0352%	0.0346%	0.0006%
33	1.8990	-14.5337	0.0373%	0.0397%	-0.0024%
34	1.8990	-14.5337	0.0395%	0.0386%	0.0009%
35	3.9241	-21.6704	0.0444%	0.0452%	-0.0008%
36	3.9241	-21.6704	0.0496%	0.0505%	-0.0009%
37	3.9241	-21.6704	0.0552%	0.0537%	0.0015%
38	3.9241	-21.6704	0.0613%	0.0572%	0.0042%

39	3.9241	-21.6704	0.0679%	0.0724%	-0.0045%
40	4.8030	-24.8823	0.0773%	0.0762%	0.0011%
41	4.8030	-24.8823	0.0870%	0.0863%	0.0008%
42	4.8030	-24.8823	0.0977%	0.1020%	-0.0043%
43	4.8030	-24.8823	0.1094%	0.1093%	0.0001%
44	4.8030	-24.8823	0.1222%	0.1198%	0.0023%
45	4.4278	-23.3771	0.1470%	0.1478%	-0.0009%
46	4.4278	-23.3771	0.1620%	0.1596%	0.0024%
47	4.4278	-23.3771	0.1781%	0.1813%	-0.0032%
48	4.4278	-23.3771	0.1955%	0.1925%	0.0030%
49	4.4278	-23.3771	0.2142%	0.2156%	-0.0014%
50	4.1410	-22.2293	0.2404%	0.2408%	-0.0004%
51	4.1410	-22.2293	0.2609%	0.2581%	0.0028%
52	4.1410	-22.2293	0.2827%	0.2840%	-0.0013%
53	4.1410	-22.2293	0.3059%	0.3110%	-0.0051%
54	4.1410	-22.2293	0.3305%	0.3265%	0.0039%
55	3.9968	-21.6054	0.3732%	0.3717%	0.0015%
56	3.9968	-21.6054	0.4010%	0.3979%	0.0031%
57	3.9968	-21.6054	0.4304%	0.4355%	-0.0051%
58	3.9968	-21.6054	0.4613%	0.4690%	-0.0078%
59	3.9968	-21.6054	0.4938%	0.4856%	0.0082%
60	3.5600	-19.7607	0.5586%	0.5519%	0.0068%
61	3.5600	-19.7607	0.5924%	0.5982%	-0.0058%

62	3.5600	-19.7607	0.6276%	0.6311%	-0.0035%
63	3.5600	-19.7607	0.6643%	0.6705%	-0.0062%
64	3.5600	-19.7607	0.7024%	0.6938%	0.0086%
65	6.5226	-32.1223	0.7459%	0.7542%	-0.0084%
66	6.5226	-32.1223	0.8237%	0.8160%	0.0076%
67	6.5226	-32.1223	0.9082%	0.9020%	0.0062%
68	6.5226	-32.1223	0.9998%	0.9964%	0.0035%
69	6.5226	-32.1223	1.0992%	1.1085%	-0.0093%
70	8.5669	-40.7731	1.2489%	1.2429%	0.0060%
71	8.5669	-40.7731	1.4091%	1.4133%	-0.0042%
72	8.5669	-40.7731	1.5871%	1.6011%	-0.0140%
73	8.5669	-40.7731	1.7843%	1.7721%	0.0122%
74	8.5669	-40.7731	2.0027%	2.0027%	0.0000%
75	9.9262	-46.6209	2.2914%	2.2914%	0.0000%
76	9.9262	-46.6209	2.6091%	2.6230%	-0.0139%
77	9.9262	-46.6209	2.9652%	2.9440%	0.0211%
78	9.9262	-46.6209	3.3635%	3.3578%	0.0057%
79	9.9262	-46.6209	3.8081%	3.8216%	-0.0135%
80	11.1740	-52.0616	4.4179%	4.4262%	-0.0083%
81	11.1740	-52.0616	5.0588%	5.0563%	0.0026%
82	11.1740	-52.0616	5.7804%	5.7449%	0.0355%
83	11.1740	-52.0616	6.5906%	6.6322%	-0.0416%
84	11.1740	-52.0616	7.4980%	7.4866%	0.0114%

85	9.8419	-46.1100	8.7912%	8.8715%	-0.0803%
86	9.8419	-46.1100	9.8095%	9.7302%	0.0793%
87	9.8419	-46.1100	10.9246%	10.9904%	-0.0658%
88	9.8419	-46.1100	12.1430%	11.8547%	0.2884%
89	9.8419	-46.1100	13.4708%	13.7016%	-0.2308%

APPENDIX E1:

FORECAST 2006 – 2015 (MALES)

Exponential trend: $y = b \times e^{a \times x}$

Age-group	b	a	2006	2007	…	2014	2015
0	0.0054	-0.021	0.43%	0.42%		0.36%	0.35%
1	0.0005	-0.025	0.04%	0.04%		0.03%	0.03%
2	0.0003	-0.061	0.02%	0.02%		0.01%	0.01%
3	0.0002	-0.028	0.02%	0.02%		0.01%	0.01%
4	0.0002	-0.018	0.01%	0.01%		0.01%	0.01%
5	0.0001	-0.010	0.01%	0.01%		0.01%	0.01%
6	0.0002	-0.045	0.01%	0.01%		0.01%	0.01%
7	0.0001	-0.020	0.01%	0.01%		0.01%	0.01%
8	0.0002	-0.031	0.01%	0.01%		0.01%	0.01%
9	0.0001	-0.020	0.01%	0.01%		0.01%	0.01%
10	0.0001	-0.058	0.01%	0.01%		0.00%	0.00%
11	0.0001	-0.027	0.01%	0.01%		0.01%	0.01%
12	0.0002	-0.026	0.01%	0.01%		0.01%	0.01%
13	0.0002	-0.049	0.01%	0.01%		0.01%	0.01%
14	0.0002	-0.043	0.01%	0.01%		0.01%	0.01%

15	0.0003	-0.033	0.02%	0.02%		0.02%	0.01%
16	0.0005	-0.039	0.03%	0.03%		0.02%	0.02%
17	0.0006	-0.053	0.04%	0.03%		0.02%	0.02%
18	0.0011	-0.058	0.06%	0.05%		0.04%	0.03%
19	0.0010	-0.051	0.06%	0.06%		0.04%	0.04%
20	0.0011	-0.062	0.06%	0.05%		0.03%	0.03%
21	0.0010	-0.049	0.06%	0.06%		0.04%	0.04%
22	0.0010	-0.046	0.06%	0.06%		0.04%	0.04%
23	0.0010	-0.053	0.06%	0.05%		0.04%	0.04%
24	0.0010	-0.043	0.06%	0.06%		0.04%	0.04%
25	0.0009	-0.034	0.06%	0.06%		0.05%	0.04%
26	0.0009	-0.029	0.06%	0.06%		0.05%	0.05%
27	0.0009	-0.036	0.06%	0.06%		0.05%	0.04%
28	0.0009	-0.039	0.06%	0.06%		0.04%	0.04%
29	0.0010	-0.044	0.06%	0.06%		0.04%	0.04%
30	0.0009	-0.036	0.06%	0.06%		0.05%	0.05%
31	0.0010	-0.032	0.07%	0.07%		0.05%	0.05%
32	0.0010	-0.038	0.07%	0.07%		0.05%	0.05%
33	0.0011	-0.035	0.07%	0.07%		0.06%	0.05%
34	0.0012	-0.038	0.08%	0.07%		0.06%	0.06%
35	0.0013	-0.039	0.08%	0.08%		0.06%	0.06%
36	0.0015	-0.047	0.09%	0.08%		0.06%	0.06%
37	0.0015	-0.045	0.09%	0.09%		0.07%	0.06%

38	0.0017	-0.041	0.11%	0.10%		0.08%	0.07%
39	0.0019	-0.044	0.12%	0.11%		0.08%	0.08%
40	0.0021	-0.044	0.13%	0.13%		0.09%	0.09%
41	0.0023	-0.043	0.15%	0.14%		0.10%	0.10%
42	0.0026	-0.039	0.17%	0.16%		0.12%	0.12%
43	0.0028	-0.035	0.19%	0.19%		0.15%	0.14%
44	0.0030	-0.032	0.21%	0.21%		0.17%	0.16%
45	0.0035	-0.034	0.24%	0.23%		0.18%	0.18%
46	0.0037	-0.026	0.28%	0.27%		0.22%	0.22%
47	0.0039	-0.020	0.32%	0.31%		0.27%	0.26%
48	0.0043	-0.018	0.35%	0.34%		0.30%	0.30%
49	0.0047	-0.020	0.38%	0.37%		0.32%	0.32%
50	0.0052	-0.020	0.42%	0.41%		0.36%	0.35%
51	0.0057	-0.020	0.46%	0.45%		0.39%	0.38%
52	0.0061	-0.016	0.51%	0.51%		0.45%	0.44%
53	0.0068	-0.020	0.55%	0.54%		0.46%	0.45%
54	0.0075	-0.019	0.61%	0.60%		0.53%	0.52%
55	0.0081	-0.017	0.67%	0.66%		0.59%	0.58%
56	0.0088	-0.018	0.72%	0.71%		0.62%	0.61%
57	0.0098	-0.022	0.77%	0.75%		0.64%	0.63%
58	0.0106	-0.019	0.86%	0.84%		0.73%	0.72%
59	0.0117	-0.021	0.93%	0.91%		0.79%	0.77%
60	0.0128	-0.022	1.01%	0.98%		0.84%	0.83%

61	0.0145	-0.027	1.07%	1.04%		0.86%	0.83%
62	0.0159	-0.027	1.18%	1.15%		0.95%	0.93%
63	0.0177	-0.031	1.26%	1.23%		0.99%	0.96%
64	0.0195	-0.033	1.36%	1.32%		1.05%	1.01%
65	0.0218	-0.033	1.51%	1.46%		1.16%	1.12%
66	0.0241	-0.035	1.63%	1.57%		1.23%	1.19%
67	0.0271	-0.038	1.78%	1.71%		1.31%	1.26%
68	0.0300	-0.038	1.96%	1.89%		1.44%	1.39%
69	0.0323	-0.037	2.15%	2.08%		1.60%	1.55%
70	0.0351	-0.034	2.43%	2.35%		1.86%	1.79%
71	0.0384	-0.032	2.71%	2.63%		2.10%	2.04%
72	0.0413	-0.029	3.01%	2.92%		2.39%	2.32%
73	0.0448	-0.027	3.32%	3.23%		2.67%	2.60%
74	0.0490	-0.025	3.71%	3.61%		3.02%	2.95%
75	0.0539	-0.025	4.10%	4.00%		3.36%	3.28%
76	0.0586	-0.025	4.46%	4.36%		3.66%	3.57%
77	0.0661	-0.027	4.91%	4.77%		3.95%	3.84%
78	0.0718	-0.025	5.44%	5.30%		4.44%	4.33%
79	0.0791	-0.026	5.95%	5.80%		4.83%	4.71%
80	0.0858	-0.023	6.68%	6.52%		5.56%	5.44%
81	0.0955	-0.023	7.37%	7.20%		6.11%	5.97%
82	0.1039	-0.021	8.27%	8.10%		7.00%	6.86%
83	0.1150	-0.021	9.16%	8.98%		7.77%	7.61%

84	0.1249	-0.019	10.17%	9.98%		8.76%	8.60%
85	0.1383	-0.020	11.05%	10.82%		9.38%	9.19%
86	0.1517	-0.020	12.16%	11.92%		10.36%	10.15%
87	0.1671	-0.021	13.26%	12.98%		11.21%	10.97%
88	0.1792	-0.018	14.73%	14.47%		12.77%	12.54%
89	0.1960	-0.018	16.15%	15.87%		14.03%	13.78%

APPENDIX E2:

FORECAST 2006 – 2015 (FEMALES)

Age-group	b	a	2006	2007	…	2014	2015
0	0.0043	-0.018	0.35%	0.35%		0.31%	0.30%
1	0.0004	-0.035	0.03%	0.03%		0.02%	0.02%
2	0.0003	-0.030	0.02%	0.02%		0.01%	0.01%
3	0.0002	-0.019	0.01%	0.01%		0.01%	0.01%
4	0.0002	-0.035	0.01%	0.01%		0.01%	0.01%
5	0.0001	0.008	0.01%	0.01%		0.01%	0.01%
6	0.0001	-0.020	0.01%	0.01%		0.01%	0.01%
7	0.0001	-0.048	0.01%	0.01%		0.01%	0.00%
8	0.0001	-0.017	0.01%	0.01%		0.01%	0.01%
9	0.0001	-0.058	0.01%	0.01%		0.00%	0.00%
10	0.0001	-0.003	0.01%	0.01%		0.01%	0.01%
11	0.0001	-0.006	0.01%	0.01%		0.01%	0.01%
12	0.0001	-0.052	0.01%	0.01%		0.00%	0.00%
13	0.0001	-0.027	0.01%	0.01%		0.01%	0.01%
14	0.0001	-0.021	0.01%	0.01%		0.01%	0.01%
15	0.0002	-0.033	0.01%	0.01%		0.01%	0.01%

16	0.0003	-0.054	0.01%	0.01%		0.01%	0.01%
17	0.0003	-0.055	0.02%	0.02%		0.01%	0.01%
18	0.0004	-0.047	0.02%	0.02%		0.02%	0.02%
19	0.0004	-0.045	0.02%	0.02%		0.02%	0.02%
20	0.0004	-0.040	0.02%	0.02%		0.02%	0.02%
21	0.0003	-0.052	0.02%	0.02%		0.01%	0.01%
22	0.0003	-0.033	0.02%	0.02%		0.02%	0.02%
23	0.0003	-0.021	0.02%	0.02%		0.02%	0.02%
24	0.0003	-0.039	0.02%	0.02%		0.02%	0.02%
25	0.0003	-0.043	0.02%	0.02%		0.02%	0.01%
26	0.0003	-0.031	0.02%	0.02%		0.02%	0.02%
27	0.0004	-0.040	0.02%	0.02%		0.02%	0.02%
28	0.0004	-0.052	0.02%	0.02%		0.02%	0.01%
29	0.0004	-0.033	0.03%	0.03%		0.02%	0.02%
30	0.0004	-0.019	0.03%	0.03%		0.03%	0.03%
31	0.0004	-0.034	0.03%	0.03%		0.02%	0.02%
32	0.0005	-0.039	0.03%	0.03%		0.02%	0.02%
33	0.0005	-0.033	0.04%	0.04%		0.03%	0.03%
34	0.0006	-0.041	0.04%	0.04%		0.03%	0.03%
35	0.0007	-0.041	0.04%	0.04%		0.03%	0.03%
36	0.0007	-0.041	0.05%	0.05%		0.03%	0.03%
37	0.0009	-0.049	0.05%	0.05%		0.03%	0.03%
38	0.0010	-0.048	0.06%	0.05%		0.04%	0.04%

39	0.0010	-0.037	0.07%	0.07%		0.05%	0.05%
40	0.0011	-0.040	0.07%	0.07%		0.05%	0.05%
41	0.0013	-0.041	0.08%	0.08%		0.06%	0.06%
42	0.0014	-0.032	0.10%	0.09%		0.08%	0.07%
43	0.0016	-0.037	0.11%	0.10%		0.08%	0.08%
44	0.0017	-0.032	0.12%	0.12%		0.09%	0.09%
45	0.0019	-0.025	0.14%	0.14%		0.12%	0.11%
46	0.0020	-0.018	0.16%	0.16%		0.14%	0.14%
47	0.0022	-0.018	0.18%	0.18%		0.16%	0.15%
48	0.0024	-0.022	0.19%	0.19%		0.16%	0.16%
49	0.0026	-0.016	0.22%	0.22%		0.19%	0.19%
50	0.0029	-0.017	0.24%	0.23%		0.21%	0.21%
51	0.0031	-0.016	0.26%	0.25%		0.23%	0.22%
52	0.0033	-0.015	0.28%	0.28%		0.25%	0.25%
53	0.0036	-0.015	0.30%	0.30%		0.27%	0.27%
54	0.0039	-0.015	0.33%	0.32%		0.29%	0.29%
55	0.0040	-0.008	0.37%	0.37%		0.35%	0.34%
56	0.0043	-0.007	0.40%	0.39%		0.37%	0.37%
57	0.0046	-0.004	0.44%	0.44%		0.43%	0.42%
58	0.0049	-0.002	0.48%	0.48%		0.47%	0.47%
59	0.0053	-0.008	0.49%	0.49%		0.46%	0.46%
60	0.0059	-0.008	0.54%	0.53%		0.50%	0.50%
61	0.0063	-0.009	0.57%	0.57%		0.53%	0.53%

62	0.0071	-0.015	0.60%	0.59%		0.54%	0.53%
63	0.0080	-0.022	0.63%	0.62%		0.53%	0.52%
64	0.0090	-0.029	0.66%	0.64%		0.52%	0.50%
65	0.0100	-0.031	0.71%	0.69%		0.56%	0.54%
66	0.0112	-0.032	0.79%	0.76%		0.61%	0.59%
67	0.0125	-0.033	0.87%	0.84%		0.67%	0.65%
68	0.0140	-0.033	0.97%	0.94%		0.74%	0.72%
69	0.0154	-0.031	1.09%	1.06%		0.86%	0.83%
70	0.0171	-0.031	1.22%	1.18%		0.96%	0.93%
71	0.0190	-0.028	1.39%	1.35%		1.11%	1.07%
72	0.0216	-0.030	1.56%	1.51%		1.23%	1.19%
73	0.0237	-0.027	1.76%	1.71%		1.41%	1.38%
74	0.0268	-0.029	1.95%	1.89%		1.55%	1.50%
75	0.0303	-0.027	2.24%	2.18%		1.80%	1.75%
76	0.0344	-0.027	2.54%	2.48%		2.04%	1.99%
77	0.0388	-0.028	2.85%	2.77%		2.28%	2.21%
78	0.0438	-0.027	3.25%	3.17%		2.62%	2.55%
79	0.0484	-0.023	3.77%	3.69%		3.15%	3.08%
80	0.0543	-0.019	4.40%	4.32%		3.78%	3.71%
81	0.0624	-0.020	5.03%	4.93%		4.29%	4.21%
82	0.0696	-0.018	5.71%	5.61%		4.94%	4.85%
83	0.0788	-0.016	6.59%	6.48%		5.78%	5.68%
84	0.0885	-0.016	7.42%	7.30%		6.53%	6.43%

85	0.0990	-0.014	8.52%	8.40%		7.63%	7.53%
86	0.1118	-0.015	9.52%	9.38%		8.46%	8.34%
87	0.1267	-0.015	10.69%	10.53%		9.45%	9.31%
88	0.1415	-0.015	12.05%	11.87%		10.71%	10.56%
89	0.1563	-0.011	13.82%	13.67%		12.64%	12.50%

APPENDIX F1:

THE GREEKS (CALL OPTION)

The values are based on following shifts in the respective underlying variable: *Delta* and *Gamma* +/- 50 bps steps in spot mortality rate, *Vega* +15 bps in volatility (for each spot level), *Rho* +5 bps (for each spot level), *Theta* -1 month (for each spot level)

Spot	Delta	Gamma	Vega	Rho	Theta
3.00%	0.2153	-	0.0013	0.0011	-0.0003%
3.50%	0.2455	0.0605	0.0046	0.0042	-0.0025%
4.00%	0.2813	0.0715	0.0110	0.0118	-0.0055%
4.50%	0.3221	0.0817	0.0194	0.0245	-0.0080%
5.00%	0.3655	0.0869	0.0292	0.0430	-0.0097%
5.50%	0.4099	0.0887	0.0384	0.0660	-0.0108%
6.00%	0.4528	0.0858	0.0456	0.0916	-0.0113%
6.50%	0.4931	0.0806	0.0500	0.1170	-0.0116%
7.00%	0.5315	0.0769	0.0519	0.1422	-0.0116%
7.50%	0.5676	0.0721	0.0514	0.1655	-0.0109%
8.00%	0.5999	0.0646	0.0489	0.1881	-0.0104%
8.50%	0.6290	0.0582	0.0451	0.2078	-0.0099%

9.00%	0.6552	0.0524	0.0401	0.2253	-0.0093%
9.50%	0.6784	0.0464	0.0349	0.2397	-0.0086%
10.00%	0.6988	0.0408	0.0302	0.2508	-0.0078%
10.50%	0.7167	0.0357	0.0253	0.2603	-0.0069%
11.00%	0.7323	0.0312	0.0210	0.2677	-0.0063%

APPENDIX F2:

THE GREEKS (PUT OPTION)

Spot	Delta	Gamma	Vega	Rho	Theta
3.00%	-0.6817	-	0.0013	-0.2983	0.0020%
3.50%	-0.6514	0.0605	0.0045	-0.2948	0.0016%
4.00%	-0.6157	0.0715	0.0109	-0.2869	0.0000%
4.50%	-0.5748	0.0817	0.0194	-0.2739	-0.0027%
5.00%	-0.5314	0.0869	0.0291	-0.2550	-0.0049%
5.50%	-0.4870	0.0887	0.0383	-0.2317	-0.0060%
6.00%	-0.4441	0.0859	0.0455	-0.2058	-0.0065%
6.50%	-0.4037	0.0808	0.0499	-0.1801	-0.0068%
7.00%	-0.3656	0.0762	0.0518	-0.1545	-0.0063%
7.50%	-0.3295	0.0724	0.0512	-0.1309	-0.0054%
8.00%	-0.2971	0.0647	0.0488	-0.1079	-0.0050%
8.50%	-0.2680	0.0582	0.0449	-0.0879	-0.0046%
9.00%	-0.2418	0.0524	0.0399	-0.0701	-0.0041%
9.50%	-0.2186	0.0464	0.0347	-0.0553	-0.0035%
10.00%	-0.1982	0.0408	0.0300	-0.0439	-0.0028%
10.50%	-0.1803	0.0357	0.0251	-0.0341	-0.0020%

11.00%	-0.1647	0.0312	0.0208	-0.0263	-0.0015%