Economic Pricing of Mortality-Linked Securities: A Tâtonnement Approach

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Abstract

In previous research on pricing mortality-linked securities, the no-arbitrage approach is often used. However, this method, which takes market prices as given, is difficult to implement in today's embryonic market where there are few traded securities. In this article, we approach the pricing problem from a different angle by considering methods that are more related to fundamental economic concepts. Specifically, we treat the pricing work as a Walrasian tâtonnement process, in which prices are determined through a gradual calibration of supply and demand. Such a pricing framework provides with us a supply curve for the investor and a demand curve for the hedger. From these curves we can tell if there will be any trade between the counterparties, and if there will, at what price the mortality-linked security will be traded. We illustrate the pricing framework with a hypothetical mortality-linked security and mortality data from the US population.

Keywords: Longevity risk; Mortality bonds; Stochastic mortality

1 Introduction

Many financial institutions are exposed to the risk of unexpected changes in human mortality. The risk is two-sided. On the one hand, life insurers paying death benefits will suffer an economic loss if actual rates of mortality are in excess of those expected,

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due to catastrophic events such as a severe outbreak of influenza and a major terrorist attack. This side of the risk is referred specifically to as mortality risk. On the other hand, pension plan sponsors, as well as insurance companies providing retirement annuities, are subject to longevity risk, that is, the risk that people outlive their expected lifetimes. For these institutions, the longer people live, the greater the period of time over which retirement income must be paid and, hence, the larger the financial liability.

An unexpected change in mortality rates will affect all policies in force. Therefore, as opposed to the random variations between lifetimes of individuals, it cannot be diversified away by increasing the size of the portfolio. Reinsurance is one possible solution to the problem, but its capacity is usually limited. Alternatively, the risk may be naturally hedged or reduced through balancing products. For example, an insurance company may sell life insurance to the same lives who are buying life annuities. The resulting combination would then reduce the company's exposure to future changes in mortality, consequently permitting a reduction of capital reserves held in respect of mortality or longevity risk. However, this strategy, as Cox and Lin (2007) pointed out, may be cost prohibitive and may not be practical in some circumstances.

In recent years, financial markets have also produced solutions by providing a variety of securities with payoffs tied to certain mortality or longevity indexes. In December 2003 we saw the first mortality bond, issued by Swiss Re. The principal of the bond would have been reduced if there had been a catastrophic mortality event during the life of the bond, therefore allowing Swiss Re to reduce some of its exposure to extreme mortality risk. Then in November 2004, BNP Paribas and the European Investment Bank (EIB) announced a 25-year longevity bond, which was intended for UK pension funds with exposures to longevity risk. This bond took the form of an annuity bond with annual coupon payments tied to the realized survival rates for some English and Welsh males. Other financial institutions such as Goldman Sachs and JP Morgan have also issued mortality-linked securities. We refer readers to Blake et al. (2008), Coughlan (2009) and Zhou and Li (2010) for further details.

A fundamental question in the study of mortality-linked securities is how to place a value on them. In previous research on pricing mortality-linked securities, the no-arbitrage approach is often used. Generally speaking, to implement the no arbitrage approach, the first step is to estimate the distribution of future mortality rates in the real-world probability measure. Then the real-world distribution is transformed to its risk-neutral counterpart, on the basis of the actual prices of mortality-linked securities we observe in the market. Finally, the price of a mortality-linked security can be calculated by discounting, at the risk-free interest rate, its expected payoff under the identified risk-neutral probability measure. Note that this approach takes actual prices as given. As we see from the following discussion, the need of market prices makes the approach difficult to implement in today's embryonic market.

One way to implement the no arbitrage approach is to use a stochastic mortality model, which is, at the very beginning, defined in the real-world measure and fitted to past data. The model is then calibrated to market prices, yielding a risk-neutral mortality process from which security prices are calculated. For instance, Cairns et al. (2006) calibrate a two-factor mortality model to the price of the BNP/EIB longevity bond, which is, as of this writing, the only long-term longevity security with pricing information available in the public domain. The resulting risk-neutral mortality process contains two market prices of risk, λ_1 and λ_2, one for each stochastic factor. With only one longevity bond price, they cannot be uniquely identified. As a result, an arbitrary assumption, for example, $\lambda_1 = \lambda_2$, must be made before any pricing work can be performed.

We may also make use of a distortion operator such as the Wang transform (Wang, 1996, 2000, 2002) to create a risk-neutral measure, under which mortality-linked securities can be priced. The Wang transform was first applied to mortality-linked securities by Lin and Cox (2005), and subsequently by other researchers including Denuit et al. (2007) and Dowd et al. (2006). Unless a very simple mortality model is assumed, parameters in the distortion operator are not unique if we are not given sufficient market price data. For example, when Chen and Cox (2009) used their extended Lee-Carter model with transitory jump effects to price a mortality bond, they were required to estimate three parameters in the Wang transform. To solve for the three parameters, Chen and Cox assumed that they were equal, but such an assumption is not easy to justify.

Recently, some researchers, for example, Li (2010) and Li and Ng (2010), have implemented the no arbitrage approach by a method called canonical valuation. This method identifies a risk-neutral measure by minimizing the Kullback-Leibler information criterion, subject to market price constraints. It can be applied without making the arbitrary decisions needed in the aforementioned methods, even if we are given only the market price of the BNP/EIB bond. However, a few problems still remain.

In particular, using a product that is very much bond-like is prone to distortions in the identification of pure longevity risk premia. One may doubt if the resulting risk-neutral measure is appropriate for pricing products with different liquidity profiles. For similar reasons, one may also question if the identified risk-neutral measure is applicable to securities that are linked to other reference populations.

In this paper, we approach the pricing problem from a different angle by considering a tâtonnement approach, an approach that is based on the most fundamental economic concept: demand and supply. The tâtonnement approach is highly transparent, since by working on the demand and supply from different economic agents, we know where the price of a mortality-linked security comes from. It also spares us from an arbitrary choice of a risk-neutral probability measure and other problems associated with the no-arbitrage approach when there is a lack of market price data. The tradeoff is that we need to impose more structure than in the no-arbitrage approach. For example, we have to specify a utility function for each party involved in the trade of a mortality-linked security.

In more detail, the method we propose models the trade between two economic agents, one of which suffers mortality or longevity risk and issues a mortality-linked security to offset the risk, and the other of which invests in the mortality-linked security, possibly for earning a risk premium. It is assumed that, given a price, both agents maximize their expected terminal utility by altering their demand or supply of the security. The estimated price of the security is the price at which the demand and supply are equal, that is, the market clears. On top of the estimated price, our pricing framework provides us with a pair of demand and supply curves. These curves can tell us the optimal quantity of a mortality-linked security to be traded. They also indicate how the supply and demand of the security will evolve with respect to a change in price. This piece of information is particularly useful when we analyze a new security that has never been traded.

The remainder of this paper is organized as follows. In Section 2, we introduce the idea of a tâtonnement process. Given this idea, we set up our pricing framework in a single-period setting, and then extend it to a multi-period one. In Section 3, we use our proposed framework to price a hypothetical mortality-linked security. The assumed mortality model, the resulting prices, and a comparison with the results from the work of Chen and Cox (2009) are presented. In Section 4 we perform sensitivity tests on different assumptions we have made in the pricing process. In

Section 5, we further generalize our pricing framework by allowing trades between the counterparties before the mortality-linked security matures. We detail in this section the required sequential decision process and an algorithm for implementing the process. Finally, in Section 6, we conclude the paper with some suggestions for further research.

2 A Tâtonnement Approach

2.1 The Idea

The idea of tâtonnement in an exchange economy was first proposed by Walras (1874).[1] A Walrasian tâtonnement process assumes that there exists a Walrasian auctioneer who matches supply and demand in a market with perfect information and no transaction costs. There exists a fictitious auctioneer who matches supply and demand from different economic agents. The agents' behaviors emerge from utility maximization, subject to budget constraints. The auctioneer cries a price, and the agents act to the price by determining how much they would like to offer (supply) or purchase (demand). Transactions only take place at equilibrium price, which equilibrates supply and demand. Otherwise, the price is lowered if there is an excess supply, or raised if there is an excess demand, until an equilibrium price is reached.

The theory of tâtonnement has different interpretations. Some economists view it as a dynamic theory of the equilibrating behavior of real competitive markets, while some treat it as a mathematical solution to the equations of general equilibrium. We refer interested readers to Goodwin (1951), Jaffé (1981) and Walker (1987) for extensive discussions on how a tâtonnement may be interpreted. Note that in a tâtonnement process, the equilibrium price might not exist, and if it exists, it might not be unique. The existence and uniqueness of a tâtonnement equilibrium price have been studied by researchers including Arrow and Hurwicz (1958, 1960) and Arrow et al. (1959). Recently, the idea of tâtonnement has received much attention in the areas of operations research and computer science. For example, Cole and Fleischer (2008) analyzed fast-converging tâtonnement algorithms for one-time and ongoing market problems.

[1] This seminal work of Walras (1874) was translated to English by William Jaffe in 1954. The translated version is cited as Walras (1954).

In what follows, we will formulate the pricing of a mortality-linked security as a tâtonnement process. We will first present the formulation in a single-period set-up, in which we assume that there is only one payout from the mortality-linked security in question. The single-period set-up is quite restrictive, but it allows the readers to capture the basic ideas behind our pricing framework. We will then extend it to a multi-period set-up, which is applicable to a wide variety of mortality-linked securities.

2.2 A Single-Period Set-up

We use a Walrasian tâtonnement process to model the trade of a mortality-linked security between two economic agents, Agents A and B. Suppose that Agent A has a life contingent liability that is due at time 1. We denote this amount by $f(q_1)$, which is a deterministic function of q_1, the mortality index for a certain reference population at time 1. In this connection, Agent A can be a life insurer which has some death benefits due at time 1, or a pension plan provider which has some living benefits due at time 1. At time 0, q_1 is not known and is governed by an underlying stochastic process.

To mitigate its exposure to mortality or longevity risk, Agent A sells a mortality-linked security maturing in one year. The payout from one unit of this security at time 1 is $g(q_1)$, which is also a deterministic function of q_1. The other economic agent, Agent B, is an investor who might invest in the mortality-linked security, possibly for earning a risk premium.

Following the specification of a tâtonnement process, we suppose that there exists an imaginary auctioneer who cries a arbitrary price, say P, at the beginning. Given this price, Agents A and B then decide their supply θ^A and demand θ^B of the mortality linked security, respectively. Their decisions are based on a certain criterion. In this paper, we assume that the agents will choose a supply or demand of the security that will maximize their expected terminal utilities.

Let ω^A and ω^B be the initial wealths of Agents A and B, respectively. It is assumed that the wealth of each agent can only be used be invested in either the mortality-linked security or a bank account which yields a continuously compounding risk-free interest rate of r per annum. We allow a negative wealth, which means that the agent borrows money from a bank account and pays an interest rate of r to the bank. Other than the bank account, the mortality-linked security and the life contingent liability, there is no other sources of income or payout.

We denote the utility functions for Agents A and B by U^A and U^B, respectively. At time 0, Agent A sells θ_A units of the mortality-linked security and deposit the proceeds and its initial wealth together into its bank account. On the other hand, Agent B uses part of its initial wealth to purchase θ_B units of the security and deposits the rest of its wealth into the bank account. At time 1, the terminal wealth for Agent A would be the amount in its bank account less the payout arising from the mortality-linked security and its life contingent liability, while that for Agent B would be the amount in its bank account plus the payout from the mortality-linked security sold by the other agent.

As mentioned earlier, at time 0, each agent choose a supply or demand of the security that maximizes its expected terminal utility. In terms of the notation defined above, the set-up can be formulated as follows:

$$\text{Agent A:} \quad \sup_{\theta^A} \mathbb{E}[U^A((\omega^A + \theta^A P)e^r - \theta^A g(q_1) - f(q_1))] \tag{2.1}$$

$$\text{Agent B:} \quad \sup_{\theta^B} \mathbb{E}[U^B((\omega^B - \theta^B P)e^r + \theta^B g(q_1))] \tag{2.2}$$

Note that θ_A and θ_B are functions of the price P. We suppress the argument of these functions for simplicity.

Usually, the first guess of the price does not clear the market. If the market does not clear, the auctioneer has to adjust the price. It is obvious that the price needs to be raised if demand exceeds supply (i.e., $\theta^B - \theta^A > 0$), and vice versa. Mathematically, the $(k+1)$th update of the price can be expressed as

$$P_{k+1} = P_k + h_k(\theta^B - \theta^A), \quad k = 1, 2, \ldots, \tag{2.3}$$

where P_0 is the initial guess of the price, and h_k is a function that always has the same sign as the excess demand, $\theta^B - \theta^A$.

In our calculations, it is assumed that the auctioneer adjusts the price in a way linear to the excess demand. Specifically, we assume that $P_{k+1} = P_k + \gamma|P_k|(\theta^B - \theta^A)$, where γ is a positive real constant. Such a linear function is intuitive and is also considered by, for example, Katzner (1999), Kitti (2010) and Uzawa (1960). The constant γ has to be chosen carefully. If γ is too large, the changes in P, θ_A, and θ_B in each iteration tend to be large. This may lead us to missing the equilibrium. In contrast, if γ is too small, the adjustment process tends to be slow. Hence, there is a tradeoff between speed and accuracy. Some experiments have been done to find an appropriate value of γ. We will revisit this problem in Section 4.

It is interesting to note that we are essentially solving the equation $Z(P) = \theta^B - \theta^A = 0$ for P. If Newton's method is used to solve the equation, then the update of P will be have the same form as equation (2.3), with

$$h_k(\theta^B - \theta^A) = \left(\frac{\partial \theta^A}{\partial P}_{|P=P_k} - \frac{\partial \theta^B}{\partial P}_{|P=P_k} \right)^{-1} (\theta^B - \theta^A).$$

Since we have $\frac{\partial \theta^A}{\partial P} \geq 0$ and $\frac{\partial \theta^B}{\partial P} \leq 0$, h_k always has the same sign as $\theta^B - \theta^A$.[2] We have experimented this alternative method and found that it works for the pricing process we present in this section. Nevertheless, it is no longer applicable to the generalized pricing process which we will present in Section 5, as that pricing process involves some discretization.

Summing up, the tâtonnement process for pricing a mortality-linked security can be carried out by the algorithm below:

Algorithm 1

1. Guess a price P_0.
2. Determine the demand, θ^B, and supply, θ^A, on the basis of the current estimate of the price and the optimizing criteria specified by equations (2.1) and (2.2).
3. If $\theta^A = \theta^B$, stop. Otherwise, adjust the price using equation (2.3).
4. Repeat Steps 2 and 3.

Let P^* be the price at which the algorithm terminates. There are two possible situations. We may obtain $P^* > 0$ and $\theta^A = \theta^B$, which means that the market attains equilibrium and trade happens between the economic agents (see Figure 1, the left panel). However, it is also possible that trade will not occur. Examples of such a situation are illustrated in the middle and right panels of Figure 1.

[2] The law of demand and supply implies that $\frac{\partial \theta^A}{\partial P} \geq 0$ and $\frac{\partial \theta^B}{\partial P} \leq 0$. To have Newton's method work, we require that the partial derivatives are not both zero.

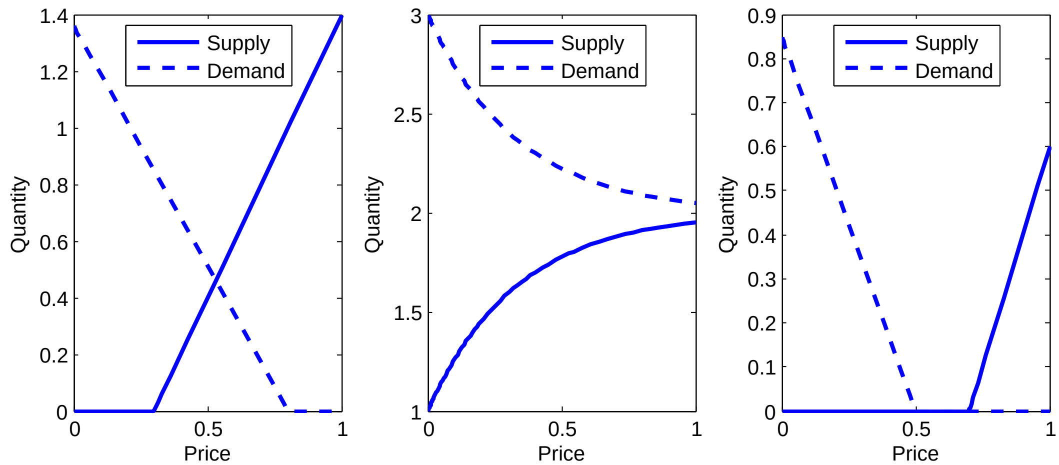

Figure 1: Possible situations when Algorithm 1 converges.

2.3 A Multi-period Set-up

We now extend the tâtonnement process for pricing a mortality-linked security to a multi-period set-up. In this set-up, we allow payments to be made before the mortality-linked security matures at time T. Denote $Q_t = (q_1, \ldots, q_t)$, where q_t is the mortality index for a certain reference population over the period of $t-1$ to t. At time 0, the values of q_t for $t > 0$ are not known and are governed by an underlying stochastic process.

Again we model the trade of a mortality-linked security between two economic agents, Agents A and B. Agent A has life contingent liabilities that are due at $t = 1, 2 \ldots, T$. The amount due at time t is $f_t(Q_t)$, where f_t is a deterministic function of Q_t. To hedge the liabilities, Agent A issues a mortality-linked security, which pays an amount of $g_t(Q_t)$ at time t, where g_t is a deterministic function of Q_t.[3] Agent B is an investor who may invest in the mortality-linked security. It receives at time t an amount of $g_t(Q_t)$ per each unit of the mortality-linked security invested.

The supply from Agent A is θ^A and the demand from Agent B is θ^B. At this stage, we do not allow the agents to trade the mortality-linked security during the term of the security. This assumption will be relaxed in the generalization we present in Section 5.

[3] We assume that f_t and g_t are functions of Q_t rather than q_t because the payouts can be path dependent.

We keep all other assumptions in the single-period set-up. Let W_t^A and W_t^B be the time-t wealths for Agents A and B, respectively. Given the assumptions we made, the wealth process for each agent can be represented as follows:

Agent A

$$\begin{aligned}
W_0^A &= \omega^A \\
W_1^A &= (W_0^A + \theta^A P)e^r - \theta^A g_1(Q_1) - f_1(Q_1) \\
W_2^A &= W_1^A e^r - \theta^A g_2(Q_2) - f_2(Q_2) \\
\vdots & \ \vdots \ \vdots \\
W_T^A &= W_{T-1}^A e^r - \theta^A g_T(Q_T) - f_T(Q_T)
\end{aligned} \tag{2.4}$$

Agent B

$$\begin{aligned}
W_0^B &= \omega^B \\
W_1^B &= (W_0^B - \theta^B P)e^r + \theta^B g_1(Q_1) \\
W_2^B &= W_1^B e^r + \theta^B g_2(Q_2) \\
\vdots & \ \vdots \ \vdots \\
W_T^B &= W_{T-1}^B e^r + \theta^B g_T(Q_T)
\end{aligned} \tag{2.5}$$

Each agent choose a demand or supply that maximizes its expected terminal utility. As such, given a price P, the supply from Agent A is $\theta^A = \text{argsup}_{\theta^A} \mathbb{E}[U^A(W_T^A)]$, while the demand from Agent B is $\theta^B = \text{argsup}_{\theta^B} \mathbb{E}[U^B(W_T^B)]$. Of course, in general, the initial guess of the price will not lead to $\theta^A = \theta^B$. We may adjust the price by using Algorithm 1, with equations (2.1) and (2.2) replaced by $\sup_{\theta^A} \mathbb{E}[U^A(W_T^A)]$ and $\sup_{\theta^B} \mathbb{E}[U^B(W_T^B)]$, respectively.

3 An Illustration

3.1 The Mortality Model

Before we implement the tâtonnement process, we need a stochastic process to model the randomness of the mortality index. We consider the Lee-Carter family, which is quite well-known in the insurance industry. The Lee-Carter model in its original form (Lee and Carter, 1992) can be expressed mathematically as

$$\ln(m_{x,t}) = \beta_x^{(0)} + \beta_x^{(1)}\kappa_t + \epsilon_{x,t}, \tag{3.1}$$

where $m_{x,t}$ denote the central death rate at age x and in year t, $\beta_x^{(0)}$ is the average level of mortality (in log scale) over time, $\beta_x^{(1)}$ is the age-specific sensitivity to the time-varying factor, κ_t, which governs the dynamics of central death rates at all ages. The error term $\epsilon_{x,t}$, which captures all remaining variations, is assumed to have no trend over both age and time dimensions. To forecast future mortality, we need to further model κ_t by a time-series process. Usually, a random walk with drift, that is,

$$\kappa_{t+1} = \kappa_t + \mu + \sigma Z_{t+1},$$

where μ and σ are constants, and $\{Z_t\}$ is a sequence of iid standard normal random variables, is used.

As shown in the work of Li and Chan (2005, 2007), the series of κ_t may be contaminated with outliers, which correspond to events such as a war and an influenza epidemic. The outliers should not be neglected in pricing mortality-linked securities, especially those for hedging extreme mortality risk. Ignoring these outliers may lead to overestimating the probability of having a catastrophic event, which may bring us a large pricing error. Therefore, rather than the original Lee-Carter model, we use an extension proposed by Chen and Cox (2009). The extension has the same structure as the specification in equation (3.1), but permits jumps in the evolution of κ_t. In particular, it assumes that, in each year, there is at most one jump event with probability p; that is,

$$N_t = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1-p, \end{cases} \tag{3.2}$$

where N_t denotes the number of jumps occurring in year t.

They further assume that the jump severity variable, Y_t, at time t is a normal random variable with mean m and standard deviation s, and that Y_t is independent of the jump frequency variable N_t. In effect, the entire stochastic process for κ_t can be expressed as follows:

$$\kappa_{t+1} = \begin{cases} \kappa_t + \mu - pm + \sigma Z_{t+1}, & \text{if } N_{t+1} = 0, \\ \kappa_t + \mu - pm + \sigma Z_{t+1} + Y_{t+1}, & \text{if } N_{t+1} = 1, \end{cases} \tag{3.3}$$

where μ and σ are constants, and Z_t is a standard normal random variable that are independent of both Y_t and N_t.

Chen and Cox (2009) fitted the extended Lee-Carter model to US mortality data from 1900 to 2003, which were provided by the National Center for Health Statistics

Parameter	Estimate
μ	-0.2172
σ	0.3872
m	-0.3062
s	2.3133
p	0.0396

Table 1: Estimates of parameters in the Lee-Carter model with jump effects.

(NCHS). The data contain age-specific death rates for age 0, age group 1-4, 10-year age groups from 5-14 to 75-84, and age group 85 and over. The resulting estimates of the parameters in equation (3.3) are displayed in Table 1. The fitted values of $\beta_x^{(0)}$ and $\beta_x^{(1)}$ can be found on p.734 of Chen and Cox (2009).

In what follows, we will price an illustrative mortality-linked security. We assume that this security is linked to the population from which the NCHS data were obtained. The extended Lee-Carter model, with parameters shown in Table 1, will be used in the tâtonnement pricing process.

3.2 Pricing a Mortality-Linked Security

We assume that the illustrative mortality-linked security is linked to a mortality index q_t, which is defined by the time-t value of the central death rate for the cohort of individuals who were aged 65 at time 0 (year 2003); that is, we set $q_t = m_{65+t,t}$.

Assume further that Agent A has sold life insurance policies which pay a total benefit $f_t(Q_t) = 50q_t$ at time t. To hedge its exposure to mortality risk, Agent A issues a mortality-linked security with \$1 face value. The security is fairly similar to the mortality bond issued by Swiss Re in December 2003. In particular, the security pays a coupon at the end of each year at a rate of r+1.5%, where r is the risk-free interest rate, which is assumed to be 3% in our baseline calculations. The principal repayment at maturity depends on the values of q_t over the term of the security. Specifically, the principal repayment is specified as follows:

$$\text{Principal Repayment} = \max\left(1 - \sum_{t=1}^{3} \text{loss}_t\right),$$

where loss_t is defined by

$$\text{loss}_t = \frac{\max(q_t - 1.1q_0, 0) - \max(q_t - 1.3q_0, 0)}{0.2q_0}.$$

In using Algorithm 1 to obtain the price of the mortality-linked security, there is a need to evaluate the expected terminal utility for each agent. The expectation can be calculated by Monte Carlo simulations as follows:

1. simulate 10 000 paths for N_t, Z_t, and Y_t;
2. calculate κ_t and q_t using the simulated paths;
3. calculate the terminal utility for each path;
4. take arithmetic average of all the simulated terminal utilities as the expected terminal utility.

We assume an exponential utility function, $U(x) = 1 - e^{-kx}$, for each agent. In the utility function, parameter k is the absolute risk aversion for all wealth levels. A larger k means that the agent is more conservative and risk averse. In a study of an insurer's optimal premium strategy, Emms and Haberman (2009) assume $k = 1.0$ for an insurer. We also assume in our baseline calculations that Agent A, an insurer, has an absolute risk aversion of $k^A = 1.0$. It is reasonable to assume that Agent A is more conservative than Agent B, because Agent A wants to hedge away its mortality risk exposure while Agent B is willing to take mortality risk in return of a risk premium. Moreover, there is a good chance that Agent B is a hedge fund, which should have a low absolute risk aversion (see, e.g., Zhu (2009)). In our baseline calculations, the assumed value of k for Agent B is $k^B = 0.5$.

Using Algorithm 1, the estimated price of the mortality-linked security is \$0.9936 and the optimal quantity of the security traded is 0.61 units. The implied risk premium is 36 basis points per annum. The risk premium may be viewed as a compensation to Agent B for taking the mortality risk.[4]

For each price level, we can calculate the demand θ_B and supply θ_A. This allows us to plot a curve of θ_B against P (the demand curve) and a curve of θ_A against P (the supply curve). The resulting demand and supply curves are shown in Figure 2.

[4]Suppose that δ be the annual risk premium. The expected payoff (in the real-world probability measure) discounted to time 0 at a rate of $r + \delta$ would be the time-0 price (\$0.9936) of the security.

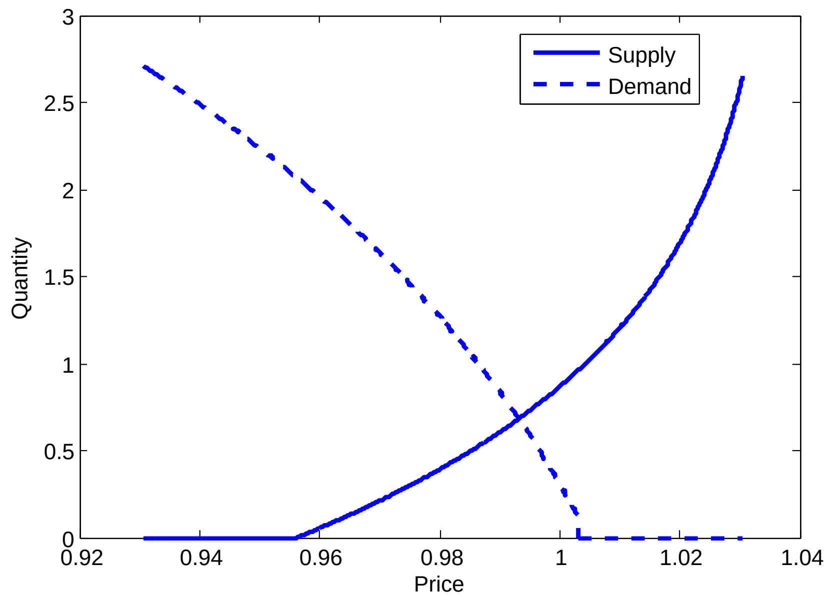

Figure 2: Supply and demand curves at time 0.

We observe that the curves intersect at one single point, giving a unique price of the security.

We also observe that the supply is 0 when price is less than \$0.9563, indicating that Agent A is not willing to sell any mortality-linked security for less than \$0.9563. When the price exceeds \$0.9563, the supply increases strictly with price. The demand has an opposite trend. It decreases with the price of the security until \$1.0031, after which it remains at 0.

3.3 Comparing with an Alternative Method

What price would an existing pricing method give to our illustrative mortality-linked security? To answer this question, we reprice the security with the pricing method in the work of Chen and Cox (2009), who use a mortality model that is completely identical to what we assume in our tâtonnement pricing process. However, rather than an economic approach, Chen and Cox use the Wang transform to identify a risk-neutral probability measure, from which the price of the mortality-linked security can be calculated.

More specifically, Chen and Cox apply the Wang transform to random variables

λ_1	4.6408	0	0.8072
λ_2	0	2.0006	0.8072
λ_3	0	0	0.8072
Price	\$0.1511	\$0.9728	\$0.8053

Table 2: Prices of the illustrative mortality-linked security implied by different market prices of risk.

Z_t, Y_t, and N_t in equation (3.3) individually to obtain the following jump mortality process in a risk-neutral probability measure:

$$\tilde{\kappa}_{t+1} = \begin{cases} \tilde{\kappa}_t + \mu - pm + \sigma \tilde{Z}_{t+1}, & \text{if } \tilde{N}_{t+1} = 0, \\ \tilde{\kappa}_t + \mu - pm + \sigma \tilde{Z}_{t+1} + \tilde{Y}_{t+1}, & \text{if } \tilde{N}_{t+1} = 1, \end{cases}$$

where $\tilde{Z}_t \sim \mathrm{N}(\lambda_1, 1)$, $\tilde{Y}_t \sim \mathrm{N}(m + \lambda_2 s, s^2)$,

$$\tilde{N}_t = \begin{cases} 1, & \text{with probability } \tilde{p}, \\ 0, & \text{with probability } 1 - \tilde{p}, \end{cases}$$

and $\hat{p} = 1 - \Phi(\Phi^{-1}(1-p) - \lambda_3)$. Here, Φ is the cumulative distribution function (cdf) for a standard normal random variable and Φ^{-1} is the inverse of the cdf for a standard normal random variable.

In the above, the unknown constants λ_1, λ_2, and λ_3 may be viewed as the market prices of risk associated with Z_t, Y_t, and N_t, respectively. Chen and Cox solve the unknowns by equating the actual price of the Swiss Re mortality bond and the price of the bond implied by the risk-neutral jump mortality process. As there are three unknowns but only one equation, there exists infinitely many possible combinations of λ_1, λ_2, and λ_3. In their calculations, Chen and Cox assume that the market prices of risk are equal. They also show in their paper the value of λ_1 when $\lambda_2 = \lambda_3 = 0$ and the value of λ_2 when $\lambda_1 = \lambda_3 = 0$, but these two sets of values are not used in their pricing work. The first three rows of Table 2 display the market prices of risk calculated in the work of Chen and Cox.

We calculate, with Monte Carlo simulations, the expected payoff from the illustrative mortality-linked security on the basis of the jump mortality process in the identified risk-neutral measure. By discounting the expected payoff at the assumed risk-free interest rate, we obtain the estimated price of the security. The prices under different assumed market prices of risk are shown in the last row of Table 2. Assuming $\lambda_1 = \lambda_2 = \lambda_3$, the method of Wang transform would give a price of \$0.8053.

However, the estimated price can be highly different if a different set of λ_1, λ_2, and λ_3 is assumed. For instance, if we assume $\lambda_2 = \lambda_3 = 0$, the estimated price would be as small as \$0.1511. The range of arbitrage-free prices is huge, and the method of Wang transform leaves us no clue to choose a price from this range.

Among the three prices in Table 2, the price based on the assumption $\lambda_1 = \lambda_3 = 0$ is the closest to the price estimated by our proposed method (\$0.9936). This is rather intuitive, as λ_2 is associated with jumps in mortality, which is exactly the risk that the security intends to hedge. It would also be interesting to see how the estimated price would change if $\lambda_1 = \lambda_2 = 0$ is assumed, as λ_3 is also associated with jumps. However, this set of market prices of risk is not provided by Chen and Cox (2009).

4 Sensitivity Tests

A few assumptions have been made in our tâtonnement pricing process. In this section, we examine how changes to these assumptions may affect the estimated price of the security in question.

4.1 Initial Wealths

Recall that we allow both agents to borrow money from the bank. This means that the initial wealth does not limit the quantity of the mortality-linked security that Agent B can purchase at time 0. Moreover, if we assume exponential utility functions for both agents, the initial wealth of each agent has no effect on the estimated price of the security.

Proposition 1. *If exponential utility functions are assumed, ω_A and ω_B have no effect on the estimated price.*

Proof. We aim to prove that ω_A and ω_B do not affect the demand and supply curves for the security. It is easy to prove by induction that

$$W_T^A = (W_0^A + \theta^A P)e^{rT} - \sum_{i=0}^{T-1} e^{(T-i-1)r}[\theta^A g_{i+1}(Q_{i+1}) + f_{i+1}(Q_{i+1})],$$

and that

$$W_T^B = (W_0^B - \theta^B P)e^{rT} + \sum_{i=0}^{T-1} e^{(T-i-1)r}\theta^B g_{i+1}(Q_{i+1}).$$

For convenience, we let

$$F(\theta^A, Q_1, Q_2, \ldots, Q_T) = \theta^A P e^{rT} - \sum_{i=0}^{T-1} e^{(T-i-1)r}[\theta^A g_{i+1}(Q_{i+1}) + f_{i+1}(Q_{i+1})]$$

and

$$G(\theta^B, Q_1, Q_2, \ldots, Q_T) = -\theta^B P e^{rT} + \sum_{i=0}^{T-1} e^{(T-i-1)r}\theta^B g_{i+1}(Q_{i+1}).$$

Given a price P, the supply θ^A is given by

$$\begin{aligned}
\text{argsup}_{\theta^A}\mathbb{E}[U^A(W_T^A)] &= \text{argsup}_{\theta^A}\mathbb{E}[e^{-k^A W_T^A}] \\
&= \text{argsup}_{\theta^A}\mathbb{E}[e^{-k^A W_0^A e^{rT}} e^{-k^A F(\theta^A, Q_1, Q_2, \ldots, Q_T)}] \\
&= \text{argsup}_{\theta^A}\mathbb{E}[e^{-k^A F(\theta^A, Q_1, Q_2, \ldots, Q_T)}],
\end{aligned}$$

which is free of W_0^A and hence ω^A. Similarly, given P, the demand θ^B is

$$\text{argsup}_{\theta^B}\mathbb{E}[U^B(W_T^B)] = \text{argsup}_{\theta^B}\mathbb{E}[e^{-k^B G(\theta^A, Q_1, Q_2, \ldots, Q_T)}],$$

which is free of W_0^B and hence ω^B. Since both initial wealths have no effect on the demand and supply curves, they do not affect the estimated price of the security. □

4.2 Risk Aversion Parameters

In the baseline calculations, we assume that the risk aversion parameters for Agents A and B are $k^A = 1.0$ and $k^B = 0.5$, respectively. We now reprice the illustrative mortality-linked security using different combinations of k^A and k^B. The estimated prices and the corresponding numbers of unit traded are displayed in Table 3.

Recall that a larger risk aversion parameter means that the agent is more risk adverse. From Table 3 we observe that at a higher k^A, more units would be sold at time 0 and the price at which the security would be sold is lower. This is because if Agent A is more risk adverse, it would have a larger intention to reduce its mortality risk exposure, leading to a higher supply of the mortality-linked security and hence a

k^A	k^B	Price	Units traded
1	0.25	\$0.9981	0.7327
1	0.5	\$0.9936	0.6116
1	1	\$0.9837	0.4590
1	2	\$0.9799	0.3061
0.5	0.5	\$0.9973	0.4172
2	0.5	\$0.9868	0.9383
4	0.5	\$0.9697	1.6052
4	2	\$0.7885	1.2177

Table 3: Prices and numbers of unit traded when different risk aversion parameters are assumed.

lower price, other things equal. On the other hand, at a higher k^B, less units would be sold at time 0 and the price at which the security would be sold is lower. This is because if Agent B is more risk adverse, it would have a smaller intention to invest in the risky security, resulting in a smaller demand and hence a lower price. From another viewpoint, if Agent B is more risk adverse, it would demand a higher risk premium for the same amount of risk. As a result, the price of the security must be lowered.

We observe that the resulting change in price is rather small if we increase k^A or k^B to two times its baseline value. However, if we increase both k^A and k^B to four times their baseline values, the reduction in the estimated price would be apparent and a much larger number of unit would be sold.

4.3 The Price Adjustment Process

In Section 2.2, we explained the role of γ in the price adjustment process. The value of γ has to be small enough. Otherwise, the price adjustment might be too big so that the equilibrium price will be missed. If γ is sufficiently small, then its value merely affects how fast the tâtonnement pricing process would converge.

To illustrate, let us consider two different values of γ. If we set $\gamma = 0.01$, the price of mortality-linked security can be found in 67 iterations. In contrast, it we set $\gamma = 0.1$, the algorithm does not converge but goes into a dead loop. The price

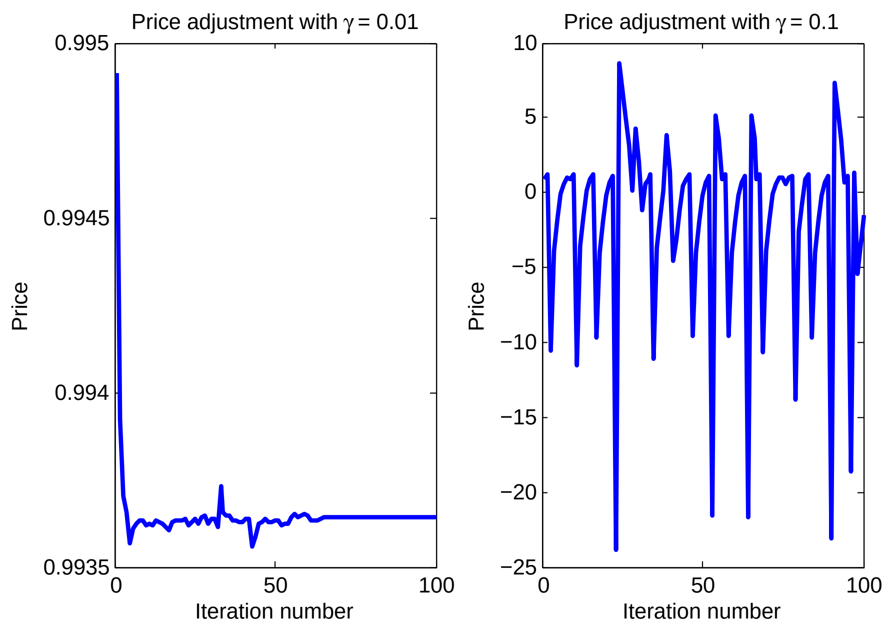

Figure 3: The price adjustment processes when $\gamma = 0.1$ and $\gamma = 0.01$.

r	0.03	0.04	0.05	0.06
Price	\$0.9936	\$0.9931	\$0.9924	\$0.9913
Units traded	0.6116	0.6175	0.6236	0.6297

Table 4: Prices and numbers of unit traded when different risk-free interest rates are assumed.

adjustment processes on the basis of the two values of γ are illustrated graphically in Figure 3.

4.4 The Risk-free Interest Rate

The risk-free interest rate plays two roles. It affects the coupons paid (the coupon rate is assumed to be $r + 1.5\%$) and also the rate at which the economic agents' wealths are accumulated. In Table 4 we show the estimated prices of the illustrative mortality-linked security under different assumed risk-free interest rates.

Table 4 indicates that the price of the illustrative mortality-linked security is not quite sensitive to the risk-free interest rate. This is possibly because when the risk-free interest rate increases, both the coupon rate of mortality bond and the return on a risk-free investment would increase. As both investment vehicles become more attractive, the increase in the demand of the mortality bond would tend to be modest.

5 Allowing Trades after Time 0

In Section 2.3, we presented the tâtonnement pricing process in a multi-period set-up. That set-up can be applied to most mortality-linked securities, but it does not permit the economic agents to trade the mortality-linked security after time 0. In this section we will generalize that set-up to allow the economic agents to sell or purchase the mortality-linked security at discrete time-steps before the security matures. This generalization may be treated as a sequential decision process.

5.1 Sequential Decision Processes

First, let us introduce a special type of sequential decision processes called Markov decision processes. In discrete Markov decision problems, decisions are made at $t = 0, 1, \ldots, T-1$. At time t, the system occupies a state. We denote the set of all possible states at time t by S_t. If, at time t, the system is in state $s_t \in S_t$, the decision maker may choose an action a_t from A_{s_t}, the set of allowable actions in state s_t. At time t, the system's state at $t+1$ is unknown and follows a certain probability distribution, which depends on s_t and a_t.

As 'Markov' indicates, the decision depends on the current state s_t only. The decision is made according to a deterministic decision rule d_t, which maps S_t to A_{S_t}. The decision rule specifies an action selection procedure in each state s_t at time t. As a result of choosing action $a_t \in A_{s_t}$, the decision maker receives an immediate reward, $r_t(s_t, a_t)$, which is a real-valued function of s_t and a_t. At maturity (time T), the decision maker receives a terminal reward of $r_T(s_T)$.

The decision rules to be used at all decision epochs compose a strategy π.[5] Specifically, a strategy π is a sequence of decision rules, that is, $\pi = (d_1, d_2, \ldots, d_{T-1})$, where $d_t \in D_t$ for $t = 1, 2, \ldots, T-1$ and for $T \leq \infty$.

[5] In the literature, π is more commonly referred to as a policy. However, to avoid any potential confusion with an insurance policy, we use the term 'strategy' instead.

Since the rewards received by the decision maker are not known before the strategy is implemented, the reward sequence is random. The decision maker's objective is to choose a strategy so that the corresponding random reward sequence is as 'large' as possible. This necessitates a method for finding an optimal strategy, which we will investigate in Section 5.2.

Some mortality-linked securities make payments that depend not only on the current but also the past values of the underlying mortality index. For such securities, a simple Markov decision process is not adequate, but we may use a less restrictive sequential decision process, which we now detail.

To explain this sequential decision process, we first represent the history of the system by the sequence of previous states and actions. The system's history, which we denote by $h_t = (s_0, a_0, \ldots, s_{t-1}, a_{t-1}, s_t)$, follows the recursion $h_t = (h_{t-1}, a_{t-1}, s_t)$. We let H_t be the set of all possible histories. Note that $H_0 = S_0, H_1 = S_0 \times A_{S_0} \times S_1$, and $H_t = S_0 \times A_{S_0} \times S_1 \times A_{S_1} \times \ldots \times S_t$.[6] Note also that the recursion $H_t = H_{t-1} \times A_{S_{t-1}} \times S_t$ holds. Here, the decision is a function of h_t and the reward function is a function of h_t and a_t. The allowable action set for each $h_t \in H_t$ is A_{h_t}. The decision process is no longer 'Markov,' because the decision rule d_t maps the entire history H_t to A_{H_t}.

Nevertheless, the less restrictive sequential decision process can be transformed into a Markov decision process. Specifically, if we set a new state variable $\hat{s}_t = h_t$, then the sequential decision process on the basis of the new state variable would have the same form as a Markov decision process. The set of all possible states at time t would become $\hat{S}_t = H_t$. As such, the theorems and algorithms developed for Markov decision processes can still be applied to transformed sequential decision processes as well.

5.2 An Optimal Strategy

Bellman (1957) provides us a simple principle to find an optimal strategy.

> *Bellman's principle of optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

[6] For sets C and D, $C \times D$ denotes the Cartesian product of C and D; that is, $C \times D = \{(c, d) : c \in C, d \in D)\}$.

Suppose that the objective is to maximize the sum of rewards from time 1 to T, that is, $\sum_{i=1}^{T-1} r_i(s_i, a_i) + r_T(s_T)$. For a Markov sequential decision process, the principle of optimality implies that $(a_0, a_1, \ldots, a_{T-1})$ is the optimal actions if

$$a_t = \operatorname*{argsup}_{a \in A_{s_t}} \{r_t(s_t, a) + \mathbb{E}[v_{t+1}(s_{t+1})|s_t, a]\},$$

where

$$v_t(s_t) = \begin{cases} \sup_{a \in A_{s_t}} \{r_t(s_t, a) + \mathbb{E}[v_{t+1}(s_{t+1})|s_t, a]\}, & t = 0, 1, \ldots, T-1, \\ r_T(s_T), & t = T. \end{cases}$$

When A_{S_t} is finite, we can replace the 'sup' in the equation above by 'max.' A mathematical definition of the principle of optimality and a detailed proof of the results above can be found in Puterman (2005).

According to the results above, we work backward from $t = T$ to find the optimal actions. At time T, we calculate all possible values of $v_T(s_T)$. We then proceed to time $T-1$. For each possible state $s_{T-1} \in S_{T-1}$, we obtain $v_{T-1}(s_{T-1})$ and the optimal action a_{T-1} by maximizing the expected value of v_T. Having completed the calculations for time $T-1$, we proceed to times $T-2, \ldots, 0$ in order. In general, the optimal action a_t and $v_t(s_t)$ at $t = 0, 1, \ldots, T-1$ can be expressed as

$$a_t = \operatorname*{argsup}_{a \in A_{s_t}} \{r_t(s_t, a) + \mathbb{E}[v_{t+1}(s_{t+1})|s_t, a]\} \tag{5.1}$$

and

$$v_t(s_t) = r_t(s_t, a_t) + \mathbb{E}[v_{t+1}(s_{t+1})|s_t, a_t], \tag{5.2}$$

respectively. The above procedure can be conveniently summarized by the following algorithm:

Algorithm 2

1. Evaluate $v_T(s_T)$ for each $s_T \in S_T$. Set $t = T$.
2. Find the optimal action a_{t-1} and $v_{t-1}(s_{t-1})$ by equations (5.1) and (5.2). Set $t = t - 1$.
3. Repeat Step 2. Stop when $t = 0$.

Algorithm 2 can also be applied to the less restrictive sequential decision process in which the distribution of s_{t+1} depends not only on s_t but also the history of the states and actions. Specifically, we replace s_t in the procedure above by $\hat{s}_t = h_t$ and the evaluation is done for each $\hat{s}_t \in \hat{S}_t$. The amount of information we include in the state variable determines the computation power we need to solve the problem. It is therefore more computationally demanding to work on the less restrictive sequential decision process.

5.3 The Generalized Pricing Process

Let θ_t^A be Agent A's short position in the mortality-linked security immediately before trade at time t. Denote by a_t^A the action taken by Agent A at time t; that is, a_t^A is the number of units of the security sold at time t. It is easy to see that $\theta_0^A = 0$ and $\theta_t^A = \theta_{t-1}^A + a_{t-1}^A$.

We define similar notation for Agent B. We let θ_t^B be Agent B's long position in the security immediately before trade at time t, and let a_t^B be the number of units of the security purchased at time t. We have $\theta_0^B = 0$ and $\theta_t^B = \theta_{t-1}^B + a_{t-1}^B$.

If the time-t payment from the security depends only on the mortality index q_t at time t, we may set the state variable as $s_t = (q_t, \theta_t)$. However, more generally, if the payment at time t also depend on values of the index before time t, then we have to use $\hat{s}_t = h_t$ as the state variable.

Again we assume that the mortality-linked security and the bank account are the only investment choices. Let $P_t(\hat{s}_t)$ be the price of the mortality-linked security at time t and at state $\hat{s}_t$. Suppose that the price process $P_0(\hat{s}_0), P_1(\hat{s}_1), \ldots, P_{T-1}(\hat{s}_{T-1})$ is known. Then the wealth processes for the two economic agents can be written as follows:

Agent A

$$
\begin{aligned}
W_0^A &= \omega^A \\
W_1^A &= (W_0^A + a_0^A P_0(\hat{s}_0))e^r - (\theta_0^A + a_0^A) g_1(Q_1) - f_1(Q_1) \\
W_2^A &= (W_1^A + a_1^A P_1(\hat{s}_1))e^r - (\theta_1^A + a_1^A) g_2(Q_2) - f_2(Q_2) \\
\vdots \;\; &\vdots \;\; \vdots \\
W_T^A &= (W_{T-1}^A + a_{T-1}^A P_{T-1}(\hat{s}_{T-1}))e^r - (\theta_{T-1}^A + a_{T-1}^A) g_T(Q_T) - f_T(Q_T) \qquad (5.3)
\end{aligned}
$$

Agent B

$$
\begin{aligned}
W_0^B &= \omega^B \\
W_1^B &= (W_0^B - a_0^B P_0(\hat{s}_0))e^r + (\theta_0^B + a_0^B)g_1(Q_1) \\
W_2^B &= (W_1^B - a_1^B P_1(\hat{s}_1))e^r + (\theta_1^B + a_1^B)g_2(Q_2) \\
\vdots \quad &\vdots \quad \vdots \\
W_T^B &= (W_{T-1}^B - a_{T-1}^B P_{T-1}(\hat{s}_{T-1}))e^r + (\theta_{T-1}^B + a_{T-1}^B)g_T(Q_T)
\end{aligned} \tag{5.4}
$$

As before, we assume that, given a price process, the agents choose their actions by maximizing their expected terminal utility. We can model this with a sequential decision process by setting $r_t(s_t, a_t)$ to 0 for $t = 0, 1, \ldots, T-1$ and $r_T(s_T)$ to the terminal utility of the agent in state s_T. We assume that agents have exponential utility functions $U^A(x) = 1 - e^{-k^A x}$ and $U^B(x) = 1 - e^{-k^B x}$.

We identify $v_t(s_t)$ in equation (5.2) for Agents A and B by $v_t^A(s_t)$ and $v_t^B(s_t)$, respectively. Note that $v_T^A(s_T) = U^A(W_T^A)$ and $v_t^B(s_T) = U^B(W_T^A)$.

Taking Agent A as an example, its optimal actions are given by

$$
\operatorname*{argsup}_{a_0, a_1, \ldots, a_{T-1}} \mathbb{E}[U^A(W_T^A)].
$$

Its optimal action at time $T-1$ is

$$
\begin{aligned}
a_{T-1}^A &= \operatorname*{argsup}_{a \in A_{\hat{s}_{T-1}}} \mathbb{E}[U^A((W_{T-1}^A + aP_{T-1}(\hat{s}_{T-1}))e^r - (\theta_{T-1}^A + a)g_T(Q_T) - f_T(Q_T))|\hat{s}_{T-1}] \\
&= \operatorname*{argsup}_{a \in A_{\hat{s}_{T-1}}} \mathbb{E}[U^A(aP_{T-1}(\hat{s}_{T-1}))e^r - (\theta_{T-1}^A + a)g_T(Q_T) - f_T(Q_T))|\hat{s}_{T-1}],
\end{aligned}
$$

which means the optimal action a_{T-1}^A depends on the past information only through Q_{T-1} and θ_{T-1}^A. If we write down the equations for $t = T-1, T-2, \ldots, 0$, we will see that the optimal action a_t^A only depends on Q_t and θ_t^A as well. Therefore, we may reduce the the state variable to $\hat{s}_t = (Q_t, \theta_t^A)$, because that would reduce the content of the state variable and hence reduce the computational effort needed.

With Algorithm 2 and the wealth processes specified by equations (5.3) and (5.4), the optimal actions for both agents can be found if we know the price process. However, the price process is not known at the outset. If we plug an arbitrary price process into Algorithm 2, the actions taken by Agents A and B are not likely to agree with each other and the market is not likely to clear. To make the market clear, we have to adjust the price process, and this can be accomplished by a tâtonnement approach, which we introduced and used in Section 2. The tâtonnement approach can be combined with a sequential decision process to solve the pricing problem.

The basic idea is to find the equilibrium price at each state successively. We begin with time $T-1$. For each possible state $\hat{s}_{T-1} \in \hat{S}_{T-1}$, the market clearing price $P_{T-1}(\hat{s}_{T-1})$ can be found by using Algorithm 1 and the optimality criterion of maximizing the expected terminal utility. At the same time, we obtain the actions a^A_{T-1} and a^B_{T-1} and the values of $v^A_{T-1}(s_{T-1})$ and $v^B_{T-1}(s_{T-1})$. Then, we repeat the procedure for times $T-2$, $T-3$, ..., 0 in order. Since there is only one state at time 0, we would be able to obtain a unique time-0 price and the corresponding optimal actions for both agents. The above procedure can be summarized by the following algorithm:

Algorithm 3

1. Evaluate $v^A_T(\hat{s}_T)$ and $v^B_T(\hat{s}_T)$. Set $t = T$.
2. For each possible state $\hat{s}_{t-1} \in \hat{S}_{t-1}$ at time $t-1$, use Algorithm 1 and equations (5.1) and (5.2) to find $P_t(\hat{s}_t)$, a^A_t, a^B_t, $v^A_t(\hat{s}_t)$, and $v^B_t(\hat{s}_t)$. Set $t = t-1$.
3. Repeat Step 2. Stop when $t = 0$. The time-0 price of the mortality-linked security is given by $P_0(\hat{s}_0)$.

5.4 A Multinomial Mortality Tree

In the sequential decision process, the system occupies a state $s_t \in S_t$ at time t. Optimal actions are determined for each state. For computational reasons, we need to keep the state space S_t discrete and finite. This means that the stochastic mortality model in Section 3.1 cannot be applied directly here, as it allows the mortality index q_t to take any non-negative real value.

To solve this problem, we construct in this section a multinomial mortality tree that is based on the stochastic mortality model in Section 3.1. Let us rewrite equation (3.3) as

$$\kappa_t - \kappa_0 = \alpha t + \sigma \sum_{i=1}^{t} Z_i + \sum_{i=1}^{t} N_i Y_i, \tag{5.5}$$

where $\alpha = \mu - pm$. Let $u_t = \kappa_t - \kappa_0$. The multinomial mortality tree we construct models the evolution of u_t over time. Given u_t and κ_0, which is known at time 0, κ_t and hence the mortality rates $m_{x,t}$ for $t > 0$ can be recovered straightforwardly.

The 'tree' (i.e., the discretized state space) for u_t is illustrated diagrammatically in Figure 4. In the diagram, $u_t(j)$ denotes the value of u_t in state j, where $j \in \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$. The construction of this tree follows from the work of Amins (1993), who developed a multinomial tree for jump diffusion option valuation in discrete time.

$$
\begin{array}{cccc}
\text{Time} = 0 & \text{Time} = 1 & \cdots & \text{Time} = t \\
 & \vdots & \cdots & \vdots \\
 & u_1(3) = \alpha + 3\sigma & \cdots & u_t(3) = \alpha t + 3\sigma \\
 & u_1(2) = \alpha + 2\sigma & \cdots & u_t(2) = \alpha t + 2\sigma \\
 & u_1(1) = \alpha + \sigma & \cdots & u_t(1) = \alpha t + \sigma \\
u_0(0) = 0 & u_1(0) = \alpha & \cdots & u_t(0) = \alpha t \\
 & u_1(-1) = \alpha - \sigma & \cdots & u_t(-1) = \alpha t - \sigma \\
 & u_1(-2) = \alpha - 2\sigma & \cdots & u_t(-2) = \alpha t - 2\sigma \\
 & u_1(-3) = \alpha - 3\sigma & \cdots & u_t(-3) = \alpha t - 3\sigma \\
 & \vdots & \cdots & \vdots
\end{array}
$$

Figure 4: The discretized state space for u_t.

Suppose u_t is at state j. If u_{t+1} is at state $j+1$ or $j-1$, we say that a local change happens between time t to $t+1$. If u_{t+1} is at other states, we say that a jump happens. As in equation (3.2), the probability of a jump is p and that of a local change is $1-p$. We need to set the transitional probabilities in such a way that the dynamics implied by the tree in Figure 4 and the process specified by equation (3.3) are approximately the same.

Conditioning on a local mortality change, the transition probability is given by

$$
\begin{aligned}
\Pr[u_{t+1} - u_t = \alpha + \sigma] &= 0.5, \\
\Pr[u_{t+1} - u_t = \alpha - \sigma] &= 0.5. \qquad (5.6)
\end{aligned}
$$

If a jump happens, we approximate the jump size distribution by assigning probability masses over non-overlapping intervals of equal width on the entire real line. Each interval is centered around one point in $\{\ldots, \alpha-3\sigma, \alpha-2\sigma, 0, \alpha+2\sigma, \alpha+3\sigma, \ldots\}$.

In equation (3.3), the jump size Y_t is assumed to follow a normal distribution with mean m and standard deviation s. Let F_Y be the cdf for Y_t. We approximate the distribution of Y_t by a probability mass function $F(i)$, where $i \neq \pm 1$, which is defined as follows:

$$\begin{aligned} F(0) &= F_Y(\alpha + 1.5\sigma) - F_Y(\alpha - 1.5\sigma), \\ F(i) &= F_Y(\alpha + (i+0.5)\sigma) - F_Y(\alpha + (i-0.5)\sigma), \; i \neq \pm 1. \end{aligned} \tag{5.7}$$

On the basis of equations (5.6) and (5.7), the unconditional transition probability can be expressed as follows:

$$\begin{aligned} \Pr[u(t+1) - u(t) = \alpha + \sigma] &= 0.5(1-p), \\ \Pr[u(t+1) - u(t) = \alpha - \sigma] &= 0.5(1-p), \\ \Pr[u(t+1) - u(t) = \alpha + i\sigma; i \neq \pm 1] &= pF(i). \end{aligned} \tag{5.8}$$

The computational burden can be reduced by using a small (finite) number of states at each time step. Specifically, at each time step, we set the maximum and minimum states to M and $-M$, respectively. This means that the total number of states at each time step is $2M+1$. The probability masses beyond the truncation points M and $-M$ are assigned to the respective truncation points.

5.5 An Example

Let us revisit the illustrative mortality-linked security in Section 3.2. We now study how the price of the security may change if we allow trades between the economic agents after time 0. As in Section 3.2, it is assumed here that $r = 3\%$, $k^A = 1$, and $k^B = 0.5$. The tâtonnement pricing process is implemented with a sequential decision process and the multinomial mortality tree in Section 5.4.

For computational reasons, the agents' positions in the mortality-linked security can only take a finite number of values. In particular, we require $\theta_t^A, \theta_t^B \in \{0, 0.01, 0.02, \ldots, 1.98, 1.99\}$ for all t. Since $\theta_t^A = \theta_{t-1}^A + a_{t-1}^A$, a_t^A must lie within the interval $(-1.99, 1.99)$ for all t. The same interval also applies to θ_t^B.

Using 15 states at each time step (i.e., $M = 7$), the estimated price of the mortality-linked security is \$0.9667. It is estimated that, at this price, 0.75 units

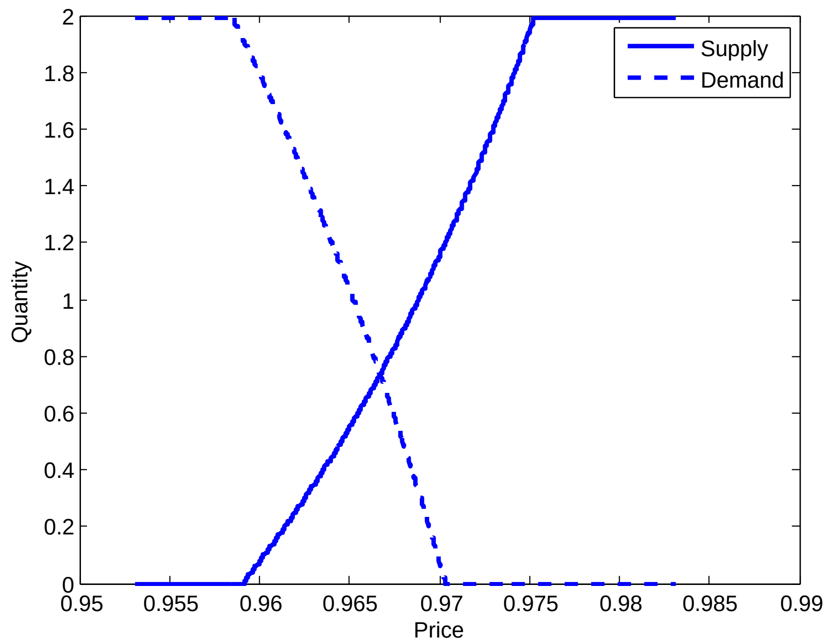

Figure 5: Demand and supply curves at time 0.

of the security would be traded. Note that this price is lower than the price estimated in Section 3.2 when we did not allow the agents to trade the security after time 0.

Figure 5 shows the demand and supply curves that are derived from the generalized pricing process which allows the agents to trade after time 0. We observe that the supply is 0 when the price is below \$0.9591, indicating that Agent A is not willing to sell the security for any price lower than \$0.9591. Then the supply curve is strictly increasing until the price reaches \$0.9752, after which the supply remains constant at 1.99 units. The upper limit of 1.99 units is because we require $\theta_t^A \in \{0, 0.01, 0.02, \ldots, 1.98, 1.99\}$ in our calculations. The demand curve has an opposite trend. It is decreasing until the price reaches \$0.9704, beyond which the demand is 0.

In Table 5 we show the estimated prices and numbers of unit traded for different choices of M. As we increase M, the values do not vary significantly, and we obtain a convergence at $M = 7$. Note that we need more computational power if we choose a larger M.

M	3	4	5	6	7	8
Price	0.9763	0.9695	0.9674	0.9669	0.9667	0.9667
Units traded at time 0	0.93	0.70	0.72	0.74	0.75	0.75

Table 5: Estimated prices for different choices of M.

6 Conclusions and Further Research

In this paper we proposed an economic method for pricing mortality-linked securities. We discussed two versions of the method. The first version, which we described in Section 2, is simple and straightforward to implement. It is highly suitable for today's market in which hedgers and investors may not find the liquidity to unwind their positions in a mortality-linked security. The second version, which was detailed in 5, permits the counterparties to unwind their positions in discrete time steps before maturity. Nevertheless, it requires more computational resources to implement, particularly if we divide the time-to-maturity into a large number of time steps.

The method we propose does not take actual market prices as given, therefore sparing us from the problems related to a lack of market price data. The advantage of not requiring market prices as input is particularly important when we price long-term longevity securities. It is not clear if annuity prices offer an adequate starting point, as it is difficult, if not impossible, to infer a pure longevity risk premium from an annuity contract. As of this writing, the BNP/EIB bond is the only long-term longevity security with pricing information available in the public domain. However, the bond did not actually trade, so the reliability of its announced price is quite questionable.

To date, most long-term longevity securities traded are bespoke securities. By a bespoke longevity security we mean the payoffs from the security are based on the actual number of survivors in the hedger's portfolio. An example is the longevity swap agreed between Babcock International and Credit Suisse in 2009. Some financial analysts believes the dominance of bespoke securities will continue in the longevity risk market over the next year or two.[7] Our pricing framework is ideal for pricing bespoke deals, as the payoff functions, f_t and g_t, and other parameters can be adapted readily to suit the actual situations of the counterparties. Further, it does not require

[7] See "Pensions Ready for Longevity Solutions," *InsuranceERM*, May 2009.

the pricing information of other mortality-linked securities, which are most likely based on different reference populations.

For pension plans and annuity providers, the reason for trading mortality-linked securities is to hedge their longevity risk exposures. In an incomplete market like the current longevity risk market, a perfect hedge cannot be formed, but an approximate hedging strategy may be constructed on the basis of a hedging objective. For example, one may construct a hedging strategy to stabilize the variability of net cash flows over a certain period of time (Cairns et al., 2008; Coughlan, 2009; Li and Hardy, 2010). If one's hedging objective is to maximize the expected utility at a certain future time, then the quantity θ^A in Section 2 can be viewed as the corresponding static hedging strategy. Further, if trades are permitted after time 0, then we may regard the actions $a_0^A, a_1^A, \ldots, a_{T-1}^A$ in Section 5 as the corresponding dynamic hedging strategy.

Our work drills down into the very fundamental economic concepts: demand and supply. The pricing framework we contribute yields a pair of demand and supply curves (Figures 2 and 5), from which we can predict if there will be any trade between the counterparties. We demonstrated empirically that, for the mortality-linked security we consider, a tâtonnement equilibrium exists and the price of the security is unique. A possible avenue of future research is to examine the necessary conditions for the existence and uniqueness of an equilibrium in the tâtonnement process for pricing mortality-linked securities. The study of such conditions may give us a better understanding of some unsuccessful longevity securities, including the BNP/EIB longevity bond, which was announced in 2004 and withdrawn for redesign in the following year, and Goldman Sachs' QxX index swap, which was launched in 2007 and withdrawn in late 2009.

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