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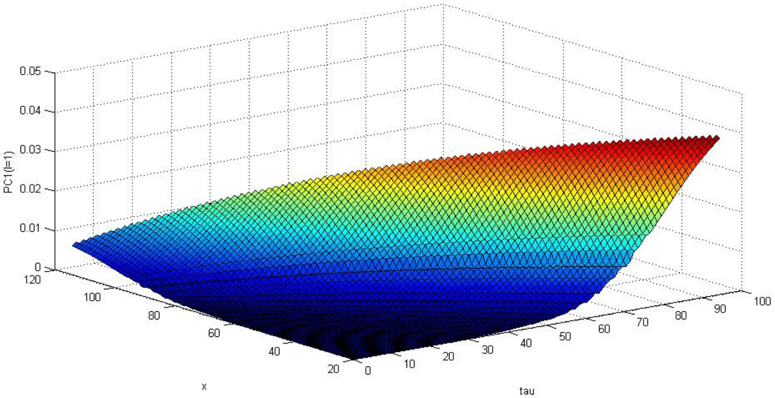

Gaussian Forward Mortality Models: Specification, Calibration, and Application

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Introduction

Specification

Calibration

Application

Conclusion

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Calibration

Application

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Motivation

Challenges for the application of stochastic mortality models in life insurance practice:

- Incompatibility with "classical life contingencies theory", which presents backbone of insurers' EDP systems
- Complexity of many of the existing approaches

⇒ Increasing discrepancy between life insurance research and actuarial practice

⇒ Potential reason for sluggish development of the longevity-linked capital market: Stochastic methods necessary to assess company's capital relief when hedging part of their exposure

⇒ ***Forward force factor models*** **as a possible solution**

Idea

$$\underbrace{{}_{\tau}p_x}_{\text{actuarial notation}} \Big(\overbrace{t}^{\text{indicates time point}} , \underbrace{t+\tau}_{\text{indicates cohort}} \Big) = \mathbb{P}(T_{x-t} > t+\tau | \mathcal{F}_t \cap \{T_{x-t} > t\})$$

$$\stackrel{!}{=} \overbrace{{}_{\tau}p_x(0, t+\tau)}^{\text{known at } t=0} \times f((\tau, x), Z_t),$$

where Z_t is a risk factor with a convenient distribution and f a known function

→ “easy" to integrate in EDP systems and, hence, to apply in practice

→ advantages in evaluation of mortality-contingent options (no nested simulations!) and "coherent" pricing of longevity derivatives (see also Dawson et al. (2010) or Duchassing and Suter (2009))

Question: How to choose f **theoretically** and **empirically consistent** (especially if Z_t is **Gaussian**)?

Forward Force of Mortality Framework

Easier to model / work with than ${}_{\tau}p_x(t, t+\tau)$:

$$\mu_t(\tau, x) = -\tfrac{\partial}{\partial \tau} \log \{{}_{\tau}p_x(t, t+\tau)\}$$

$$\left[\text{also possible:} \quad \tilde{\mu}_t(\tau, x) = -\tfrac{\partial}{\partial \tau} \text{logit} \{{}_{\tau}p_x(t, t+\tau)\} \quad \rightarrow \text{current work}\right]$$

Consider **time-homogenous diffusion-driven** models (cf. Bauer et al. (2010))

$$d\mu_t = (A\,\mu_t + \alpha)\; dt + \sigma\, dW_t$$

- $A = \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial x}\right)$
- Drift condition (Cairns et al. (2006, ASTIN)): If W is BM under $\mathbb{P}$,
  $$\alpha(\tau, x) = \sigma(\tau, x) \times \int_0^{\tau} \sigma'(s, x)\, ds$$
- Bauer et al. (2010): (μ_t) allows for a Gaussian finite-dimensional realization (FDR) iff $\sigma(\tau, x) = C(x+\tau)\, \exp\{M\,\tau\}\; N$
- **Proposition:** If $\sigma(\tau, x) = (\sigma_1(\tau, x), \ldots, \sigma_d(\tau, x))$ and each $\sigma_\nu(\tau, x)$ allows for a FDR, $1 \leq \nu \leq d$, then $\sigma(\tau, x)$ allows for a FDR.

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Principal Component Analysis

- Assume we have generation life tables at $t_j = t_0 + \Delta\, j$, $1 \leq j \leq N$. Define

$$F_j(\tau, x) = -\log\left\{ \frac{{}_{\tau+1}p_x(t_{j+1}, t_{j+1}+\tau+1)}{{}_{\tau}p_x(t_{j+1}, t_{j+1}+\tau)} \Big/ \frac{{}_{\tau+1+\Delta}p_{x-\Delta}(t_j, t_{j+1}+\tau+1)}{{}_{\tau+\Delta}p_{x-\Delta}(t_j, t_{j+1}+\tau)} \right\}$$

$$\stackrel{d}{=} \mathbb{E}\left[F_j(\tau, x)\right] + \underbrace{\int_{\tau}^{\tau+1} C(x+\nu)\, e^{M\nu}\, d\nu}_{O(\tau,x)} \times \underbrace{\int_0^{\Delta} e^{M(\Delta-s)}\, N\, dW_s}_{=Z_\Delta}$$

$\rightarrow$ Normal distributed with “known" covariance structure

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- IDEA: Use PCA to assess covariance structure empirically and then determine model $(C(x), M, N)$ that matches eigenvectors ($dim(N) = m \times d$):

$$\hat{\Sigma} = U \times \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_K) \times U' = \sum_{\nu=1}^{K} \lambda_\nu\, u_\nu\, u_\nu'$$

$$\approx \sum_{\nu=1}^{K} \lambda_\nu\, u_\nu\, u_\nu' = Cov\left(\sum_{\nu=1}^{d} u_\nu \sqrt{\lambda_\nu}\, \text{Normal}_{\nu,j}\right)$$

$$\stackrel{!}{=} Cov\left((O(\tau_i, x_i))_{1 \leq i \leq K} Z_\Delta\right)$$

$\rightarrow$ Determine model via regression on eigenvectors, can separate because of Prop. above ($d = 1$ for each principle component)

Data

- **"Pensioners Data"**: 7 Mortality tables as published by the Institute of Actuaries
- **"Population Data"**: 30 generation tables (1978-2008) compiled based on Lee-Carter method for 30 years before, England/Wales male mortality experience ($\rightarrow$ Human Mortality Database, `www.mortality.org`)

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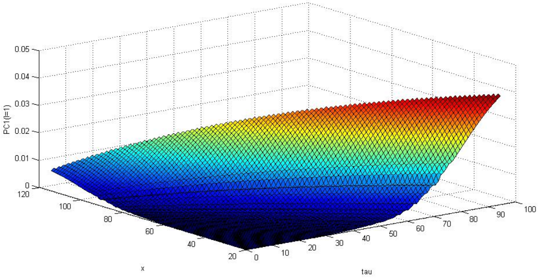
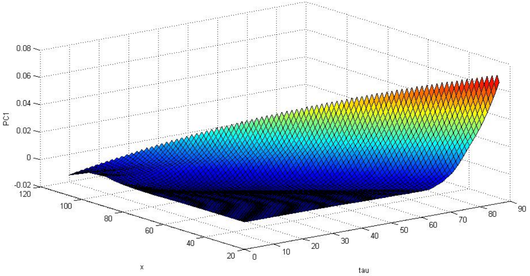
- **Results:**

	Population		**Pensioners**	
λ_1	1.2822	98.33%	0.7506	94.61%
λ_2	0.0182	1.40%	0.0290	3.66%
λ_3	0.0018	0.14%	0.0103	1.3%
λ_4	0.0007		0.0029	
λ_5	0.0005		0.0006	
λ_6	0.0005		0.0000	

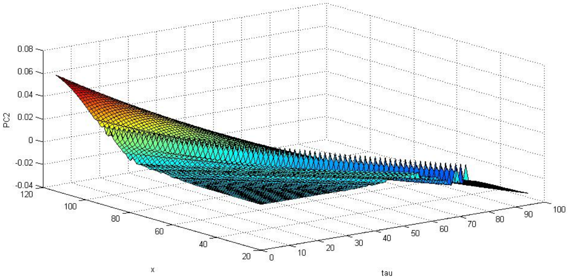
- Plot eigenvectors as function of τ and x

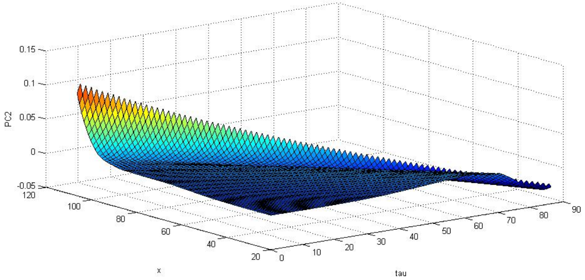
PC 1 (population and pensioner)



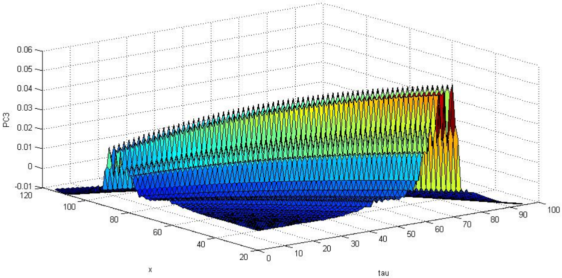


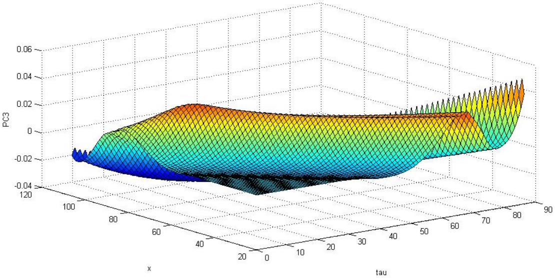
PC 2 (population and pensioner)





PC 3 (population and pensioner)





Determination of Suitable Specification

- **Method 1:** Just find suitable $C(x)$, M, N via regression. No restrictions on parameters.
  - $m = 1$ not sufficient to depict structure $\rightarrow$ dimension $m = 2$
  - Identification issues $\rightarrow$ set off-diagonals of M zero
  - Still lots of parameters
- **Method 2:** Mimic shape of "diagonal curves" using familiar examples from interest rate modeling
  - Concave shape: (PC1 and PC3)

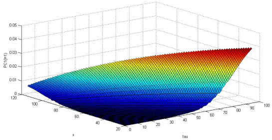
$$
\begin{aligned}
C(x+\tau) &= f(x+\tau) \times \begin{bmatrix} 0 & 1 \end{bmatrix}, M = \begin{bmatrix} -2b & -b^2 \\ 1 & 0 \end{bmatrix}, \text{and} \\
N &= \begin{bmatrix} -ab-1 \\ a \end{bmatrix} \\
\Rightarrow \sigma(\tau, x) &= C(x+\tau)\exp(M\tau)N = f(x+\tau)(a+\tau)\exp(-b\tau).
\end{aligned}
$$

  - Hump Shape: (PC2)

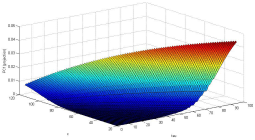
  $\rightarrow$ Difference in exponentials

PC 1 (population only)

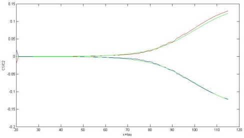
Method 1:



Eigenvector

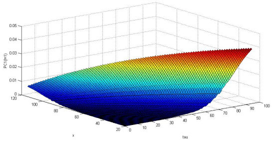


Projection

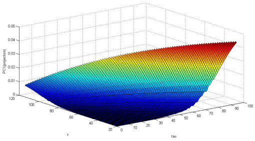


$C(x)$

Method 2:



Eigenvector



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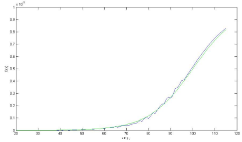
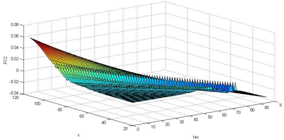


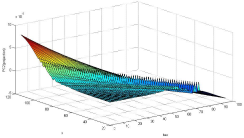
$C(x)$

PC 2 (population only)

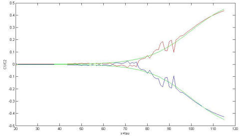
Method 1:



Eigenvector



Projection



$C(x)$

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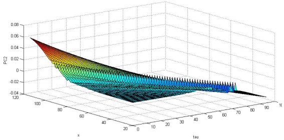
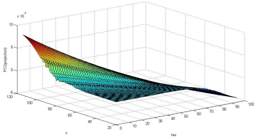


Eigenvector



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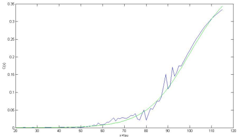
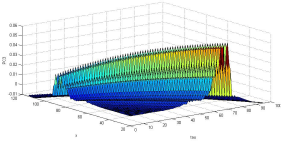


$C(x)$

PC 3 (population only)

Method 1:



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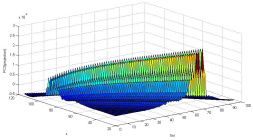

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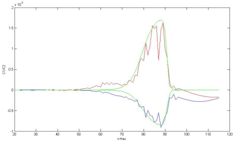

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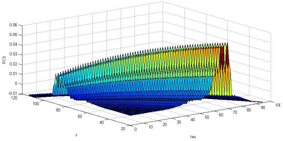

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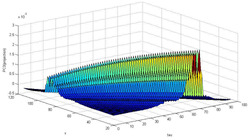

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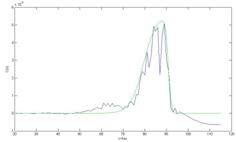

$C(x)$

Calibration

We obtain parameter values from the PCA but...

1. Relied on approximations in the process that may lead to small biases
2. Only consider second moments (Covariance matrix) although first moment depends on second moment ($\rightarrow$ Drift condition)

...so we use **numerical ML approach** (relying on known Normal distribution of $F_j(\tau, x)$) but use PCA parameters as starting values in our numerical optimization (similar to Bauer et al. (2008, APJRI)).

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Difference:

- Allow for non-systematic deviations (similar to *measurement equation* in state space approaches) $\rightarrow$ Essentially picks up "white noise" from non-considered principal components

$\Rightarrow$ Allows us to consider "large" (ca. 80) selection of age/term combinations. Not all due to numerical problems with inverting "too large" covariance matrix in likelihood function

Method 1: 24 free parameters. **Method 2**: 18 free parameters. Essentially same likelihood, so we use Method 2 results in what follows.

Economic Capital Modeling

- Stylized, newly funded life insurer. Uses equivalence principle with current yield curve and current generation life table for pricing.

$\Rightarrow$ Available Capital at time zero: $\mathrm{AC}_0 = [\mathrm{Equity}] = E$

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- Portfolio of Policies:

	x	i	$n_{x,i}^{\text{term/end/ann}}$	Benefit $B_{\text{term/end/ann}}$
Term Life	30	20	250	100, 000
	35	15	250	
	40	10	250	
	45	5	250	
Endowment	40	20	500	50, 000
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Annuities	60	(45)	250	18, 000
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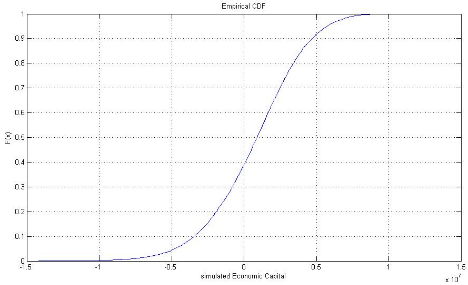
- Available Capital at time one: $\text{AC}_1 = \mathbb{E}^{\mathbb{Q}}[\text{Assets}|\mathcal{F}_1] - \mathbb{E}^{\mathbb{Q}}[\text{Liabilities}|\mathcal{F}_1]$
- One-year mark-to-market approach for calculating Economic Capital: $\text{EC} = \rho\,(\text{AC}_0 - p(0,1)\,\text{AC}_1)$, where $\rho(\cdot)$ is monetary risk measure

Preliminary Results



Without system. mortality risk:

← Distribution of portfolio loss L

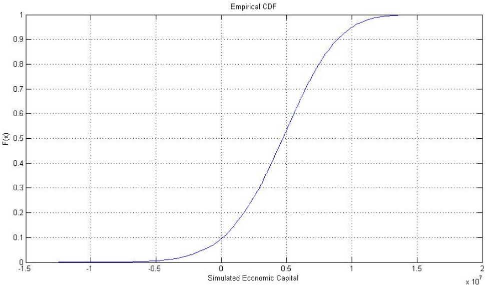
$\text{EC} = V@R_{90\%}(L) = \$4,720,600$

With system. mortality risk:

Distribution of portfolio loss $L \rightarrow$

$\text{EC} = V@R_{90\%}(L) = \$8,917,100$

<ins>Caveat:</ins> Do not consider impact of stochastic mortality on "unsystematic" risk – may explain difference in expected values.



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Conclusion

- Application of stochastic models in life insurance practice important for risk management

- One of the key issues with respect financial market in longevity risk

→ Need for tractable models that are compatible with insurer's EDP systems (based on classical actuarial theory) but rich enough to depict the risk situation accurately

- Gaussian Forward Mortality Factor Models satisfy these criteria. We address their
  - Specification (via PCA approach),
  - Calibration, and their
  - Application

  thereby providing a basically ready-to-use workhorse

Contact

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Thank you!