Stochastic Mortality Modeling with Lévy processes

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ABSTRACT

- In the classical Lee-Carter model, the mortality indices *kt* assumed to be a random walk model with drift is normally distributed.
- The assumption of normality is stacked against the hard fact.
- So, we adopt two infinitely divisible distributions—Generalized Hyperbolic (GH) and Classical Tempered Stable (CTS) distributions—to model the mortality indices.
- We show that GH or CTS distribution assumptions are better than normal distribution assumption.

Agenda

- Introduction
- The Lee-Carter Model with GH and CTS Innovations
  - The Lee-Carter model
  - Normality Test for Mortality Indices
  - Modeling Mortality indices as GH and CTS Processes
- Empirical Analysis
  - Mortality Data
  - Model Comparison
- Conclusions

Introduction

- Lee and Carter (1992)…popular.
- Brouhns, Denuit, and Vermunt (2002), Renshaw and Harberman (2003), Denuit, Devolder, and Goderniaux (2007), and Li and Chan (2007)…various modifications of the Lee-Carter model.
- Cairns, Blake, and Dowd (2006), and Cairns et al. (2009)…CBD model and its extension.
- Haberman and Renshaw (2009)… forecast age-period-cohort mortality and test for robustness.
- The above papers *don't* include mortality jumps.

Introduction

- Biffis (2005)… model asset prices and mortality dynamics.
- Luciano and Vigna (2005)… for Italian mortality data…add a jump component and fit better.
- Cox, Lin and Wang (2006)… for US and UK …geometric Brownian motion and compound Poisson …Swiss Re bond.
- Lin and Cox (2008)… Brownian motion and a discrete Markov chain with log-normal jump size distribution… incomplete market framework.
- Chen and Cox (2009)… a jump process into the Lee-Carter model.
- The above papers use diffusion processes with jump components, *finite activity* Lévy processes, to describe the dynamics of mortality rates.

Introduction

- A pure jump Lévy process can display either finite activity or infinite activity.
- The classical example of a finite-activity jump process is the compound Poisson jump process.
- Variance Gamma process … infinite activity … finite variation … a special case of CTS process.
- Normal Inverse Gaussian process … infinite activity … infinite variation … a special case of GH process.

Introduction

- Hainaut and Devolder (2008) first applies α-stable subordinators—infinite-activity strictly positive Lévy processes—to model the mortality hazard rates.
- However, in the Lee-Carter model, the first difference of mortality indices may be negative to reflect the mortality improvement.
- Unlike Hainaut and Devolder (2008), in this paper we incorporate the GH and CTS processes into the original Lee–Carter model, and use it to fit and forecast the mortality rates.

Introduction

- We select the ninety-five-year mortality data of ten countries—Demark, Finland, France, Iceland, Italy, Netherlands, Norway, Spain, Sweden and Switzerland—as the observed data.
- We fit the model to the mortality rates from 1912 to 2001 and forecast the development of the mortality curve for the consequent five years.

The Lee-Carter Model with GH and CTS Innovations

- The Lee-Carter model
- Normality Test for Mortality Indices
- Modeling Mortality indices as GH and CTS Processes

The Lee-Carter model

- Lee and Carter (1992)

$$\ln\left(m_{x,t}\right) = \alpha_x + \beta_x k_t + e_{x,t}$$

$$\sum_t k_t = 0 \quad \text{and} \quad \sum_x \beta_x = 1$$

$$\varepsilon_t \sim N\left(0, \sigma^2\right)$$

ARIMA(0,1,0)

$$k_t - k_{t-1} = \gamma + \varepsilon_t$$

Our key point

$$k_t^L - k_{t-1}^L = L_1$$

Lévy processes

GH, CTS…

Normality Test for Mortality Indices

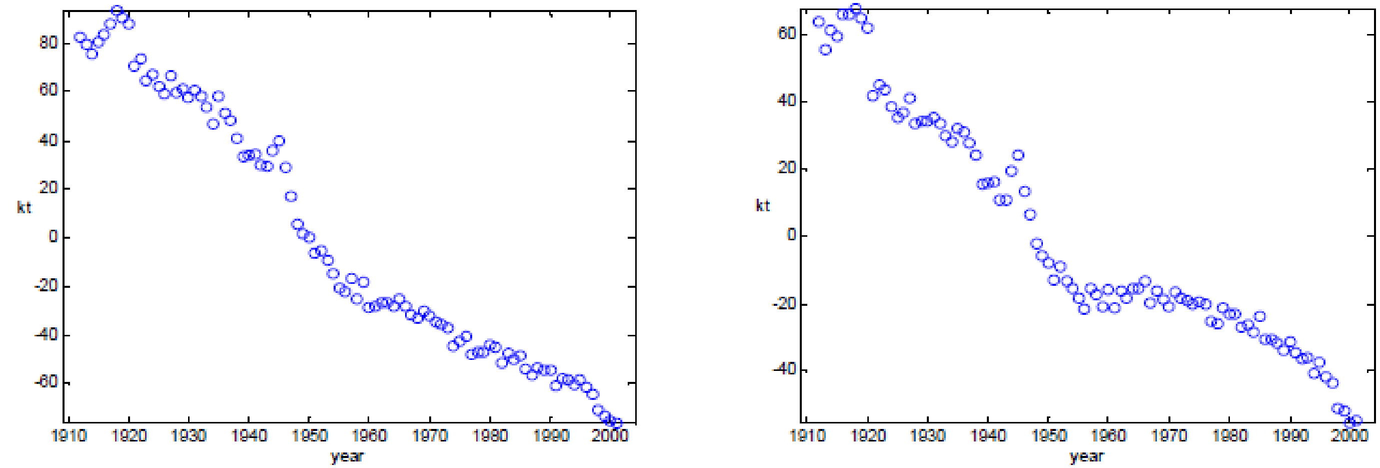

Figure 1A. Demark Female (Left) and Male (Right).

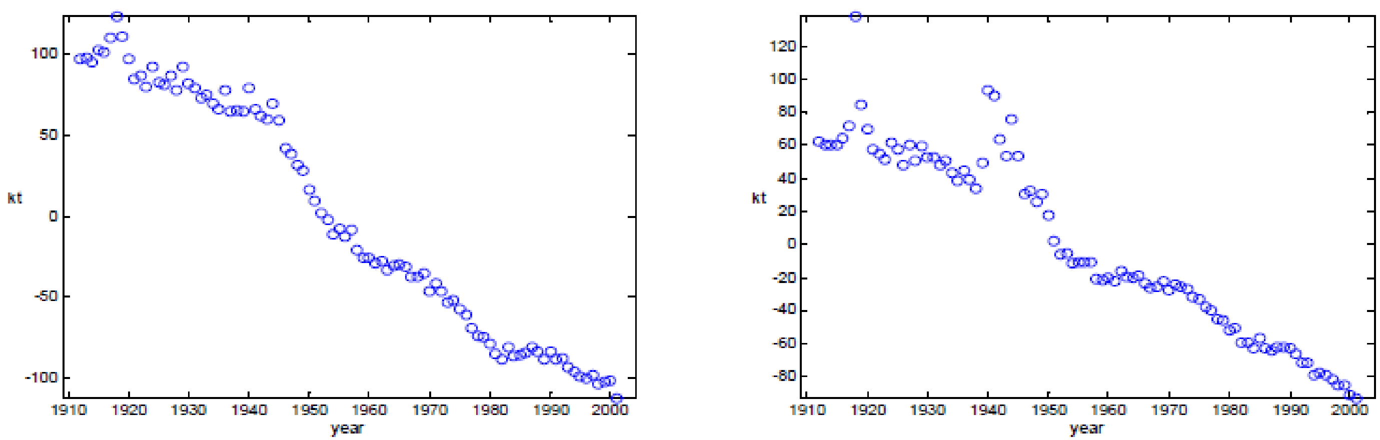

Figure 1B. Finland Female (Left) and Male (Right).

Normality Test for Mortality Indices

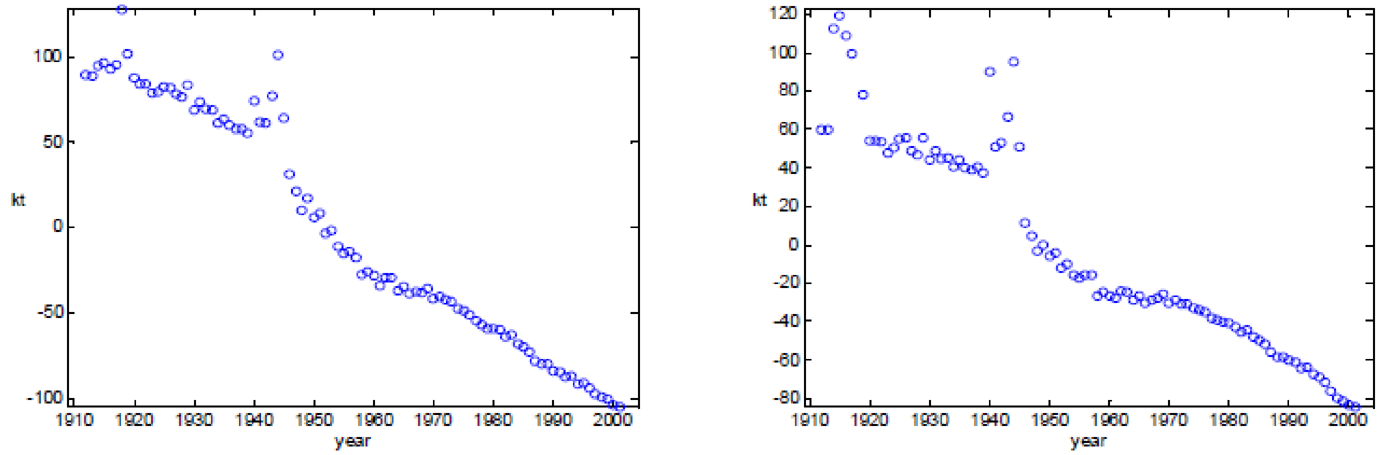

Figure 1C. France Female (Left) and Male (Right).

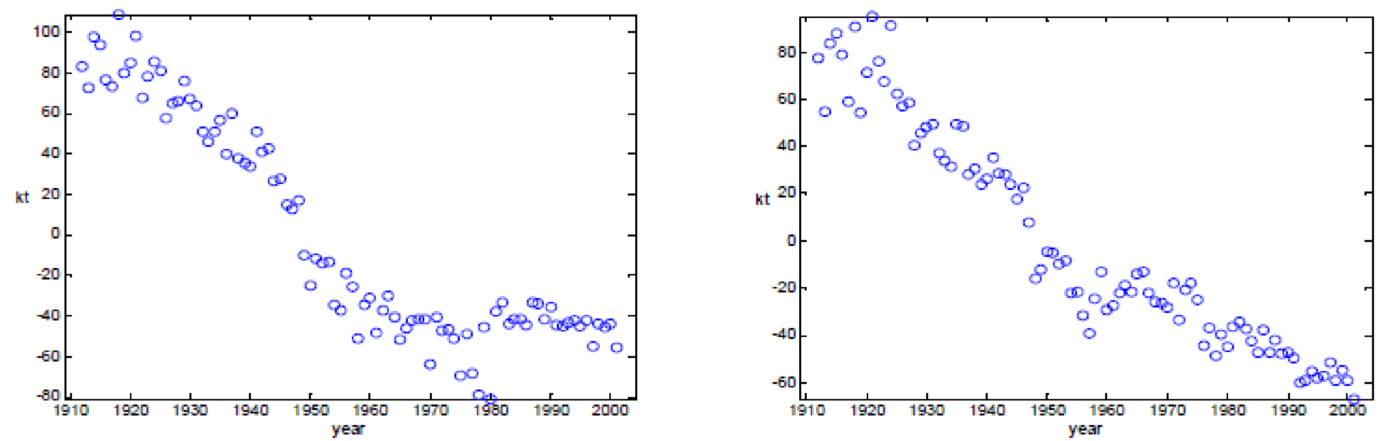

Figure 1D. Iceland Female (Left) and Male (Right).

Normality Test for Mortality Indices

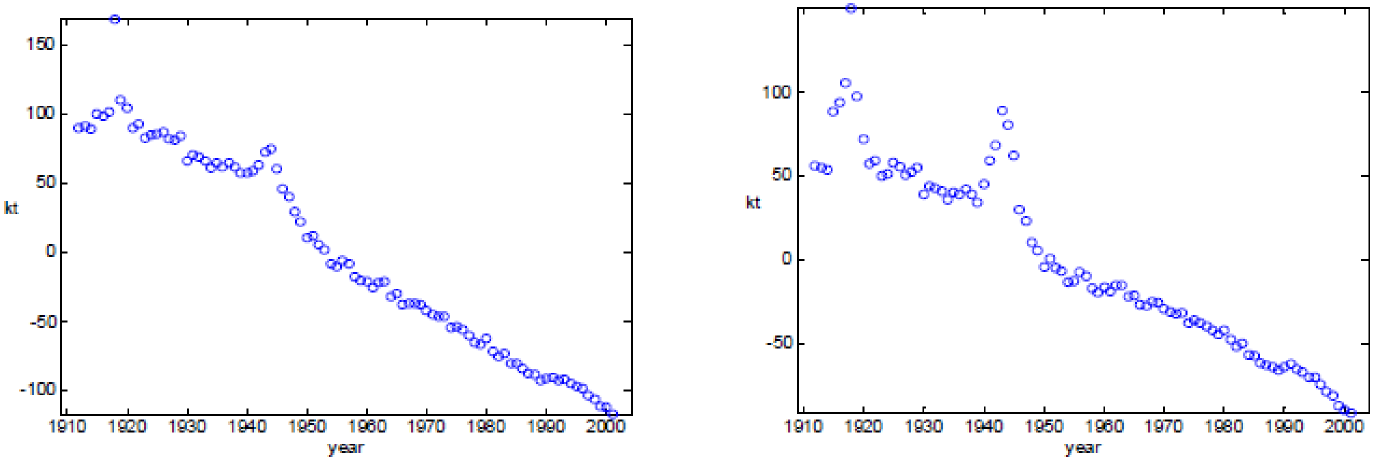

Figure 1E. Italy Female (Left) and Male (Right).

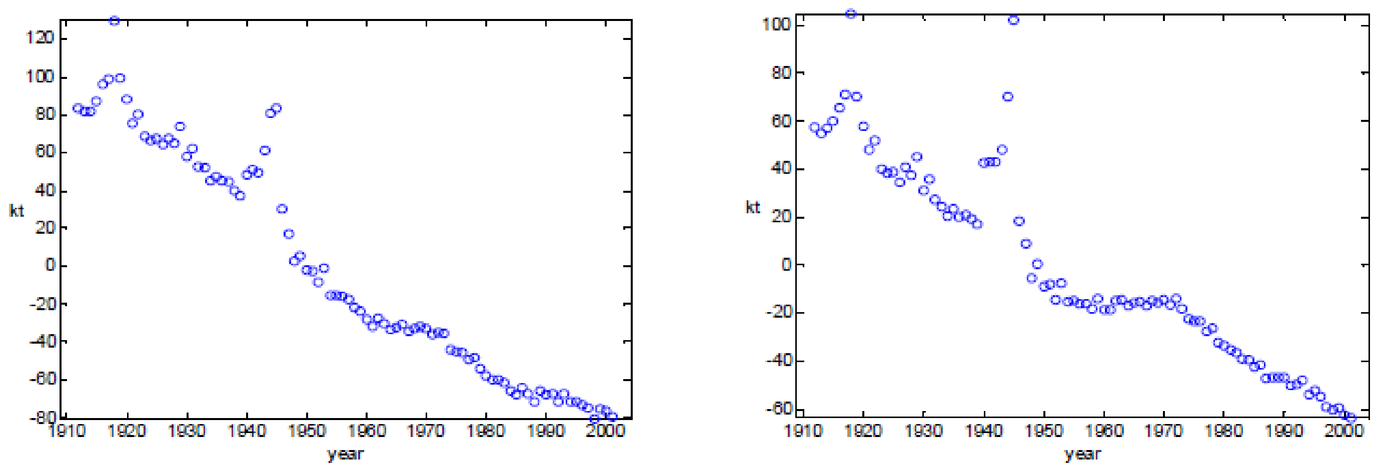

Figure 1F. Netherlands Female (Left) and Male (Right).

Normality Test for Mortality Indices

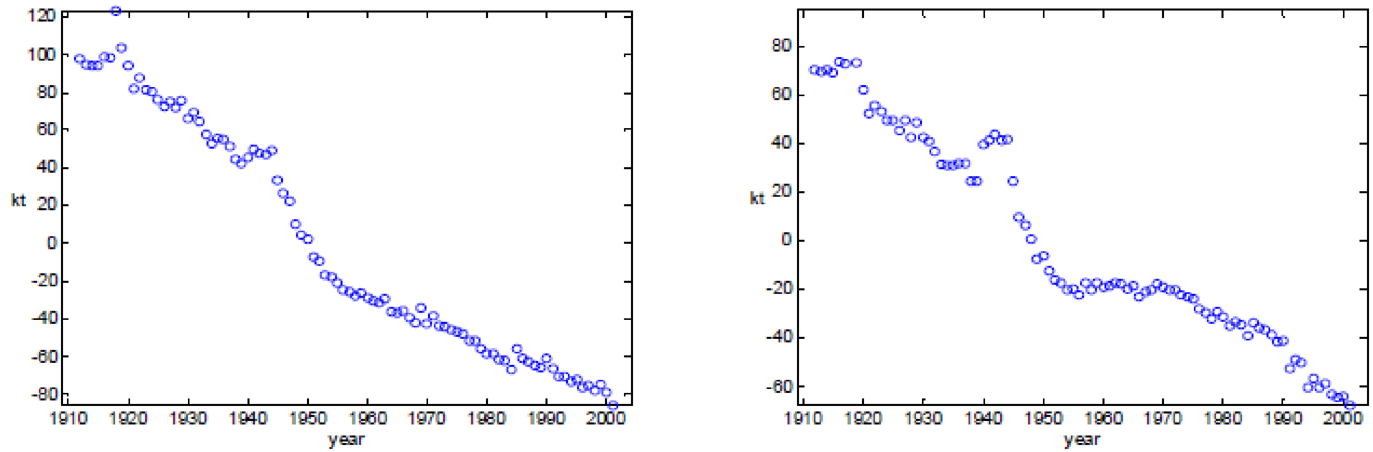

Figure 1G. Female (Left) and Male (Right).

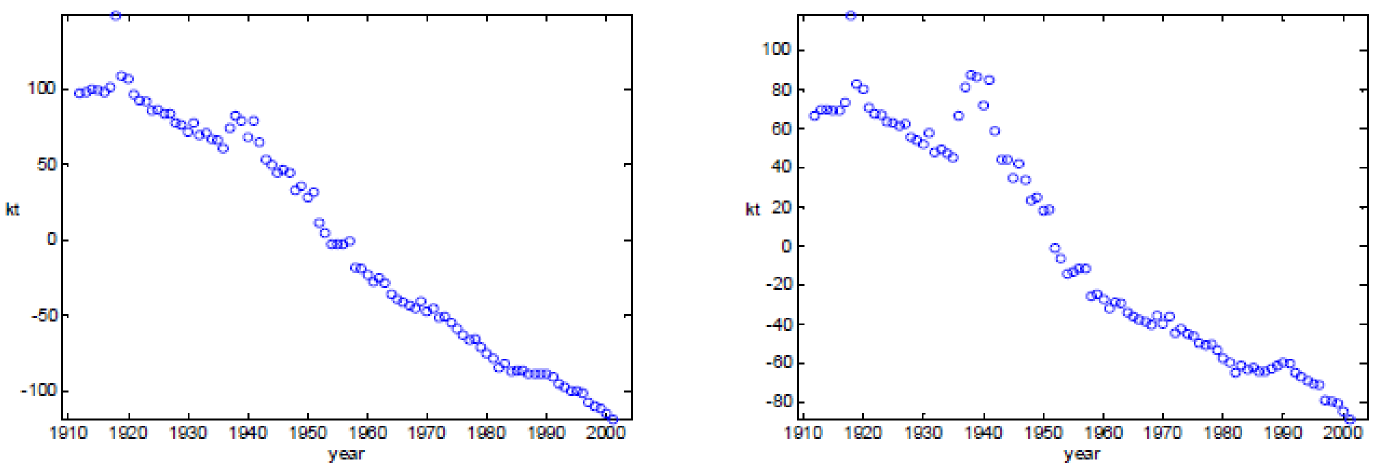

Figure 1H. Spain Female (Left) and Male (Right).

Normality Test for Mortality Indices

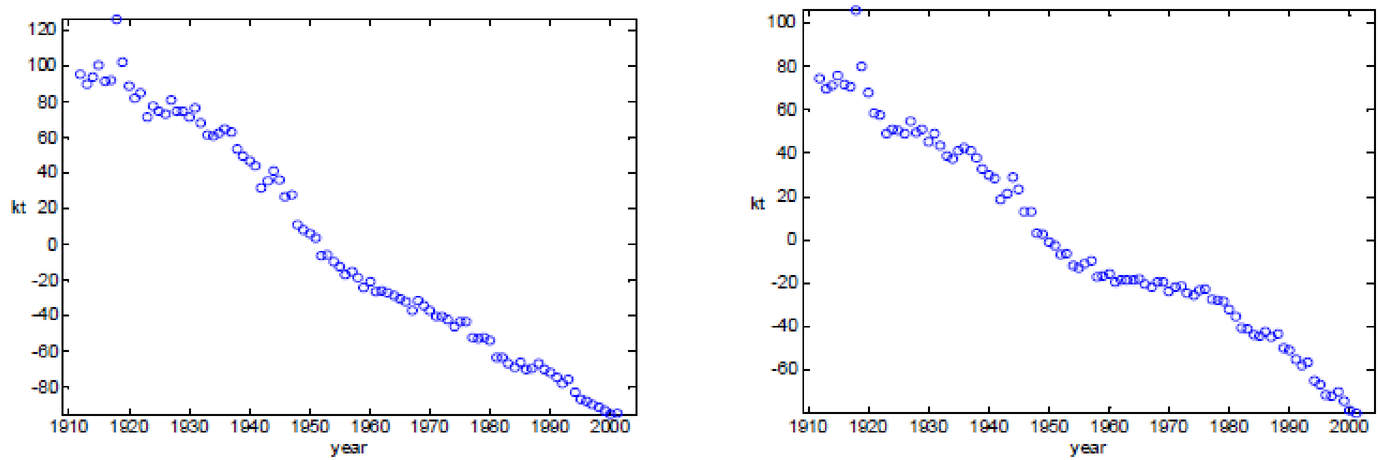

Figure 1I. Sweden Female (Left) and Male (Right).

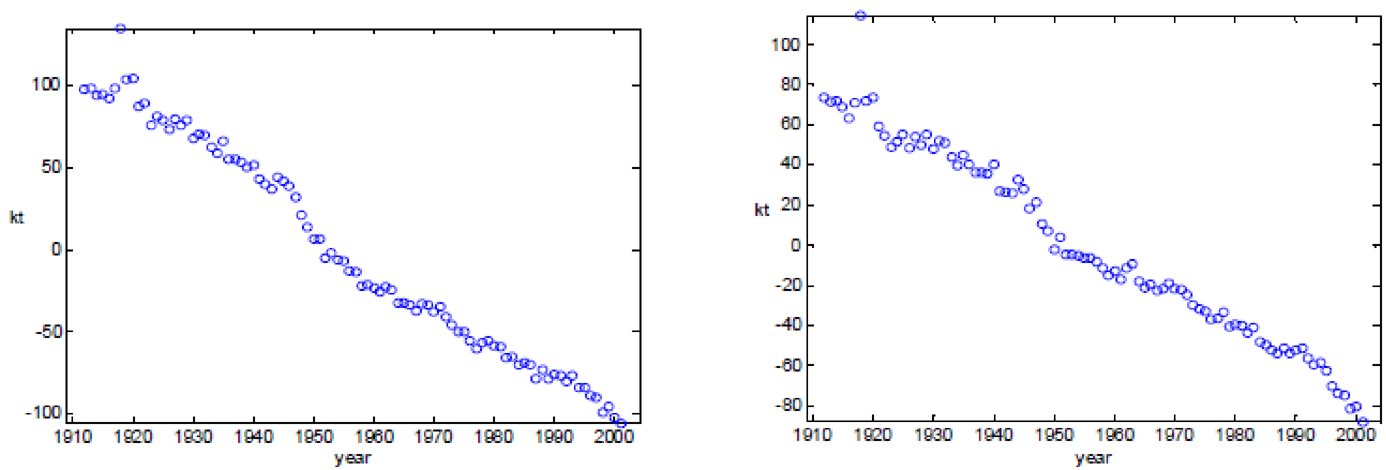

Figure 1J. Switzerland Female (Left) and Male (Right).

Normality Test for Mortality Indices

H_0 : The data are from a normal distribution.

- The JB test statistic (Jarque and Bera, 1980), a goodness-of-fit measure of departure from normality, as follows:

$$JB = n\left[\frac{s^2}{6} + \frac{(k-3)^2}{24}\right]$$

where n is the sample size; s is the sample skewness; k and is the sample kurtosis.

Normality Test for Mortality Indices

Country	Gender	P-value	Skewness	Excess Kurtosis
Denmark	Female	0.248	-0.302	0.426
	Male	<0.001	-0.738	2.783
Finland	Female	0.284	0.352	0.063
	Male	<0.001	1.476	13.001
France	Female	<0.001	-0.219	6.612
	Male	<0.001	0.462	8.203
Iceland	Female	0.083	0.327	0.776
	Male	0.063	0.025	1.095
Italy	Female	<0.001	1.418	27.125
	Male	<0.001	-0.120	9.555
Netherlands	Female	<0.001	-1.670	12.193
	Male	<0.001	-3.037	23.809
Norway	Female	<0.001	0.904	6.299
	Male	<0.001	0.126	4.906
Spain	Female	<0.001	1.312	17.559
	Male	<0.001	0.962	10.622
Sweden	Female	<0.001	1.354	11.542
	Male	<0.001	1.815	17.930
Switzerland	Female	<0.001	1.049	12.520
	Male	<0.001	0.615	19.881

Modeling Mortality indices as GH and CTS Processes

$$k_t - k_{t-1} = L_1$$

where L_1 is the Lévy process with unit time scale. When L_1 is the GH (CTS) process, the mortality index k_t follows GH (CTS) distribution.

Modeling Mortality indices as GH and CTS Processes

Processes	Characteristic Function $\phi(\omega)$
Generalized Hyperbolic model $GH(\alpha,\beta,\delta,\lambda,\mu)$	$e^{i\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\lambda/2}\frac{K_\lambda\left(\delta\sqrt{\lambda^2-(\beta+iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$
Classical Tempered Stable $CTS(\alpha,C_+,C_-,\lambda_+,\lambda_-,m)$	$\exp\left\{ium+\Gamma(1-\alpha)\left(C_+\lambda_+^{\alpha-1}-C\_\lambda_-^{\alpha-1}\right) +C_+\Gamma(-\alpha)\left((\lambda_+-iu)^\alpha-\lambda_+^\alpha\right) +C\_\Gamma(-\alpha)\left((\lambda_-+iu)^\alpha-\lambda_-^\alpha\right)\right\}$

Note: K is the modified Bessel function of the second kind, and Γ is a gamma function.

Modeling Mortality indices as GH and CTS Processes

- The GH distribution includes the NIG distribution, the hyperbolic distribution, the normal distribution, Student *t* distribution, the skew hyperbolic-*t* distribution, and the variance gamma distribution.
- The CTS distribution includes the truncated Lévy flight distribution, the CGMY distribution, the Variance Gamma distribution, the normal distribution.
- Therefore, we provides a more flexible and abundant mortality index models with GH and CTS innovations than that with normal innovation.

Empirical Analysis

- Mortality Data
- Model Comparison

Mortality Data

- Human Mortality Database (HMD)
- Ten countries: Demark, Finland, France, Iceland, Italy, Netherlands, Norway, Spain, Sweden and Switzerland.
- Age: 0 to 100.
- Fitting period: 1912 to 2001.
- Forecasting period: 2002 to 2006.

Model Comparison

- Log likelihood function (LLF)
- Akaike Information Criterion (AIC)

  AIC = (-2*LLF) + (2*NPS)

  where NPS is the effective number of parameters being estimated.
- Bayesian information criteria (BIC)

  BIC = (-2*LLF) + (NPS * log(NOS))

  where NOS is the number of observations.
- Kolmogorov–Smirnov test

  H_0 : The sample is drawn from the reference distribution.

Good-of-Fit Measures for Female

Female		LLF	AIC	BIC	KS p-value
Denmark	Normal	-281.6312	567.2624	572.2620	0.0143
	GH	-266.0407	542.0813	554.5804	**0.9954**
	CTS	**-266.0381**	**542.0762**	**554.5752**	0.9942
Finland	Normal	-317.3149	638.6298	643.6294	0.0175
	GH	**-292.8436**	**595.6872**	**608.1862**	**0.6544**
	CTS	-292.8464	595.6929	608.1919	0.6493
France	Normal	-311.0207	626.0413	631.0410	0.0027
	GH	**-289.2394**	**588.4787**	**600.9778**	**0.9225**
	CTS	-291.3043	592.6087	605.1077	0.6013
Iceland	Normal	**-323.9160**	**651.8320**	**656.8316**	0.5365
	GH	-362.3137	734.6274	747.1264	**0.9489**
	CTS	-362.3246	734.6492	747.1483	0.9385
Italy	Normal	-319.6590	643.3180	648.3176	0.0002
	GH	**-284.8698**	**579.7396**	**592.2387**	**0.8639**
	CTS	-287.3213	584.6427	597.1417	0.3910
Netherlands	Normal	-314.2988	632.5976	637.5973	0.0013
	GH	**-293.1537**	**596.3074**	**608.8064**	**0.9261**
	CTS	-293.5052	597.0104	609.5095	0.7154
Norway	Normal	-272.4171	548.8343	553.8339	0.1826
	GH	**-265.6993**	**541.3985**	**553.8976**	**0.9990**
	CTS	-265.7121	541.4241	553.9232	0.9984
Spain	Normal	-307.0227	618.0454	623.0451	0.0058
	GH	**-278.6876**	**567.3751**	**579.8742**	0.8670
	CTS	-279.8252	569.6504	582.1494	**0.9079**
Sweden	Normal	-300.1478	604.2955	609.2951	0.0830
	GH	**-271.8271**	**553.6542**	**566.1533**	0.8476
	CTS	-273.0809	556.1618	568.6609	**0.9467**
Switzerland	Normal	-301.6165	607.2331	612.2327	0.1811
	GH	**-279.7489**	**569.4978**	**581.9968**	**0.9985**
	CTS	-279.8568	569.7137	582.2127	0.9929

Good-of-Fit Measures for Male

Male		LLF	AIC	BIC	KS p-value
Denmark	Normal	-259.6505	523.3010	528.3006	0.0425
	GH	-253.6852	517.3704	529.8695	0.7456
	CTS	**-253.1297**	**516.2595**	**528.7585**	**0.7683**
Finland	Normal	-322.9167	649.8334	654.8330	0.0012
	GH	**-312.1082**	**634.2164**	**646.7155**	**0.9880**
	CTS	-313.7963	637.5927	650.0917	0.6842
France	Normal	-323.1906	650.3813	655.3809	0.0000
	GH	-297.9508	605.9016	618.4006	**0.9966**
	CTS	**-291.7267**	**593.4535**	**605.9525**	0.2486
Iceland	Normal	-323.3071	650.6143	655.6139	0.5617
	GH	**-342.3913**	**694.7827**	**707.2817**	**0.9946**
	CTS	-342.4289	694.8578	707.3568	0.9931
Italy	Normal	-320.0279	644.0558	649.0554	0.0022
	GH	-301.0792	612.1584	624.6575	0.8997
	CTS	**-286.8498**	**583.6996**	**596.1986**	**0.9018**
Netherlands	Normal	-322.3932	648.7863	653.7859	0.0002
	GH	**-293.6083**	**597.2166**	**609.7157**	**0.8486**
	CTS	-295.6467	601.2934	613.7925	0.8130
Norway	Normal	-270.8644	545.7288	550.7284	0.0022
	GH	**-261.6509**	**533.3017**	**545.8008**	**0.9998**
	CTS	-262.0067	534.0134	546.5124	0.9880
Spain	Normal	-309.9867	623.9735	628.9731	0.0039
	GH	**-286.4151**	**582.8302**	**595.3292**	**0.9921**
	CTS	-286.9939	583.9879	596.4869	0.8771
Sweden	Normal	-275.6614	555.3228	560.3224	0.0166
	GH	**-257.7139**	**525.4279**	**537.9269**	**0.8817**
	CTS	-258.2076	526.4151	538.9141	0.6671
Switzerland	Normal	-303.0029	610.0058	615.0054	0.0531
	GH	**-275.4430**	**560.8861**	**573.3851**	**0.9761**
	CTS	-276.5020	563.0040	575.5030	0.8912

Model Comparison

- The definition of the Mean Absolute Percentage Error (MAPE) is calculated as follows:

$$MAPE = 100\% \times \frac{1}{n} \sum_{i=1}^{n} \left| \frac{A_i - F_i}{A_i} \right|$$

  where A_i is historical mortality rate and F_i is the forecast mortality rate.

- Forecasting period: 2002 to 2006 … five years.
- Simulate 100,000 paths.

Percentile of MAPE of Mortality Projection for Female (Unit: %)

Female		5%	10%	Medium	90%	95%
Denmark	Normal	**3.4661**	**3.4848**	**3.5996**	**3.9596**	**4.0840**
	GH	3.4821	3.5034	3.7137	4.4009	4.6574
	CTS	3.4778	3.5005	3.7161	4.2933	4.5534
Finland	Normal	7.1603	7.2002	7.6322	8.5164	8.8146
	GH	**7.1345**	**7.1602**	**7.4194**	**8.0261**	**8.2470**
	CTS	7.1348	7.1723	7.4431	8.0841	8.3508
France	Normal	3.5614	3.5894	3.8911	5.0650	5.4992
	GH	**3.5217**	**3.5361**	**3.7007**	**4.8471**	**5.4722**
	CTS	3.5319	3.5536	3.7987	5.3770	6.3538
Iceland	Normal	10.5264	10.5812	11.0350	12.5004	13.0243
	GH	10.5170	**10.5588**	11.0394	12.6986	13.5475
	CTS	**10.5136**	10.5598	**10.9045**	**12.4920**	**13.2356**
Italy	Normal	4.1625	4.1903	4.5423	6.0183	6.6042
	GH	**4.1252**	**4.1372**	**4.2982**	**5.3761**	**5.9318**
	CTS	4.1316	4.1423	4.3559	5.8481	6.7074
Netherlands	Normal	2.9293	2.9699	3.3566	4.6307	**5.0872**
	GH	**2.8961**	**2.9166**	**3.1286**	**4.5533**	5.7357
	CTS	2.9043	2.9279	3.2173	4.7998	5.8568
Norway	Normal	**5.4941**	**5.5092**	5.5938	**5.8619**	**5.9884**
	GH	5.4961	5.5112	5.5928	5.8806	6.0660
	CTS	5.5019	5.5119	**5.5906**	5.8833	6.0096
Spain	Normal	4.2810	4.3192	4.5913	5.4238	5.8462
	GH	**4.2544**	**4.2728**	**4.4257**	**5.1299**	**5.7671**
	CTS	4.2554	4.2823	4.4859	5.4550	6.0277
Sweden	Normal	3.8128	3.8275	3.9756	**4.4927**	**4.6687**
	GH	**3.7951**	**3.8049**	3.9123	4.5567	4.9596
	CTS	3.7963	3.8056	**3.9106**	4.5457	4.8950
Switzerland	Normal	4.0601	4.0793	4.2413	4.8233	**5.0412**
	GH	**4.0472**	**4.0574**	4.1814	**4.7487**	5.1421
	CTS	4.0487	4.0617	**4.1806**	4.7575	5.2766

Percentile of MAPE of Mortality Projection for Male (Unit: %)

Male		5%	10%	Medium	90%	95%
Denmark	Normal	4.8640	4.8773	5.0096	**5.3456**	**5.4648**
	GH	4.8661	4.8789	4.9964	5.3920	5.5433
	CTS	**4.8629**	**4.8761**	**4.9925**	5.4102	5.6029
Finland	Normal	8.0899	8.1501	8.6171	10.3698	11.1702
	GH	8.0403	**8.0758**	**8.3922**	**9.9201**	11.3050
	CTS	**8.0382**	8.0908	8.4744	9.9820	**11.0628**
France	Normal	5.0559	5.1106	5.7243	7.9078	8.7535
	GH	**4.9889**	**5.0164**	**5.3147**	**6.8770**	**8.0444**
	CTS	4.9996	5.0433	5.5282	8.2538	9.3840
Iceland	Normal	11.2200	11.2890	11.6577	**12.8096**	**13.4348**
	GH	**11.2156**	11.2708	**11.6262**	12.8255	13.5292
	CTS	11.2196	**11.2807**	11.6611	12.9423	13.6455
Italy	Normal	5.2765	5.3187	5.7539	7.2911	**7.8877**
	GH	**5.2503**	**5.2719**	**5.5599**	7.4513	8.4238
	CTS	5.2513	5.2732	5.6115	**7.1392**	8.0599
Netherlands	Normal	4.0499	4.1410	4.8670	6.7219	7.3716
	GH	**3.9520**	**3.9871**	**4.3069**	**6.0929**	**7.3205**
	CTS	3.9620	4.0003	4.4576	6.8600	8.5175
Norway	Normal	6.4322	6.4506	6.5525	6.7766	**6.8710**
	GH	**6.4361**	**6.4458**	**6.5123**	**6.7655**	7.0372
	CTS	6.4327	6.4460	6.5198	6.8094	7.0229
Spain	Normal	3.7542	3.7951	4.1770	5.3553	**5.7795**
	GH	**3.7197**	**3.7376**	**3.9743**	**5.3256**	6.4212
	CTS	3.7199	3.7417	3.9897	5.4643	6.1843
Sweden	Normal	5.1920	5.2041	5.2909	5.6212	5.7940
	GH	**5.1787**	**5.1858**	**5.2427**	**5.5158**	**5.7149**
	CTS	5.1810	5.1889	5.2590	5.6478	5.9990
Switzerland	Normal	5.4998	5.5220	5.7623	6.5570	6.8685
	GH	**5.4769**	**5.4857**	**5.5864**	**6.1664**	**6.5336**
	CTS	5.4783	5.4886	5.6292	6.4505	7.2256

Conclusions

- We make the first attempt to incorporate two infinity-activity pure jump Lévy processes — Generalized Hyperbolic (GH) and Classical Tempered Stable (CTS) Processes — into Lee-Carter model respectively.
- The results consistently indicate a preference for the mortality indices with GH or CTS distributions over those with normal distribution.
- In this paper, we do not provide the valuation and hedging of mortality securities. Using the Esscher transform, future research may focus on how to price and hedge the mortality-linked security such as Swiss Re mortality bond issued in 2003.

Thank you for your attention.