Pricing of Swiss Re mortality bond: an Equilibrium Approach

Jeffrey T. Tsai
National Tsing Hua University

Larry Y. Tzeng
National Taiwan University

Sep 2010

Price of Swiss Re Mortality Bond

- Question 1: Swiss Re Mortality Bond over compensated the investors?
    - the offered rate is LIBOR+135 bps
    - Cox, Lin and Wang (2006): using the market price of risk of propriety catastrophe bond, and their spread is 45 bps
    - Lin and Cox (2008): their spread is 39 bps.
- Question 2: Existing valuation approaches in literature

Existing valuation approaches in literature

- Wang transform (Wang, 2000, 2002)
  - provides a distortion operator that transforms the underlying distribution to an equivalent risk-adjusted one, applied to discount the expected cash flows with a risk-free rate.
  - Lin and Cox (2005) and Cox, Lin and Wang (2006)
- Arbitrage-free pricing (Cairns, Blake, and Dowd 2006b)
  - if the market is arbitrage free, at least one risk-neutral measure Q exists for using to calculate fair prices.
  - they estimate the market price of longevity risk is constant from the longevity risk premium, implied by the proposed issue price of the EIB/BNP longevity bonds.
  - Milevsky and Promislow (2001), Dahl (2004) employ similar assumptions and treatments.

MLCCs: Swiss Re mortality bonds

- Mortality bonds (Swiss Re, 2002)
  - three-year and the amount was $400 million.
  - bondholders receive coupons quarterly at a rate of three-month LIBOR plus 135 bps.
  - the principle is not full protected
    - the principle depends on the mortality index q_t weighted by five countries' mortality experiences.
    - if q_t exceeds the 130% of the 2002 level, q_0 the principal will reduce 5% for every 1% raise in the index. If q_t exceeds 150% of q_0, the principal is exhausted.

MLCCs: Swiss Re mortality bonds

- The percentage loss of principal in year t

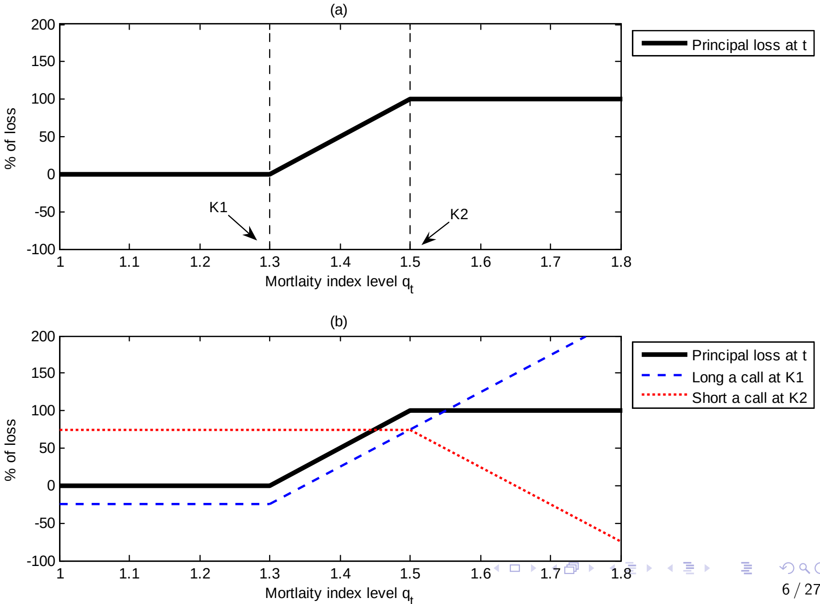
$$L_t = \begin{cases} 0\% & \text{if } q_t < 1.3q_0 \\ (q_t - 1.3q_0)/0.2q_0 & \text{if } 1.3q_0 \leq q_t \leq 1.5q_0 \\ 100\% & \text{if } q_t > 1.5q_0 \end{cases} \quad (1)$$

- The principal paid back to the bondholders at maturity is:

$$B_T = 400 \text{ million} \times Max\left(100\% - \sum_t L_t,\ 0\right) \quad (2)$$

where $t = 1, 2, 3$ for year 2003, 2004 and 2005.

Figure 1. Terminal loss of Principal and Bull Spread (Revised from Blake, Cairns and Dowd, 2006a)



Loss of Principal (Bull spread)

- The loss percentage at $t = 1$ is

$$L_1 = Max\left(\frac{q_1 - K_1}{K_2 - K_1}, 0\right) - Max\left(\frac{q_1 - K_2}{K_2 - K_1}, 0\right) \qquad (3)$$

$$= \frac{Max\left(q_1 - K_1, 0\right) - Max\left(q_1 - K_2, 0\right)}{K_2 - K_1}$$

- If we find values of this bull spread, we find L_1. The losses L_2 and L_3 can also be obtained by replacing q_1 with q_2 and q_3.

We calculate the value of terminal principal at time 0 with some proper discounting:

$$B_0 = 400 \text{ million} \times PV\left[Max\left(100\% - \sum_t L_t,\ 0\right)\right] \qquad (4)$$

An Approximation Method

Lin and Cox (2008) and Chen and Cox (2009) propose an approximation method on $\sum_t L_t$ as following:

$$\sum_t L_t = \frac{Max\left(q_{\max} - K_1, 0\right) - Max\left(q_{\max} - K_2, 0\right)}{K_2 - K_1} \qquad (5)$$

where $q_{\max} = Max(q_1, q_2, q_3)$.

- The probability of two mortality catastrophes at sequence-years are rare, so they choose the maximum value of q_t for a representative mortality level.
- This simplification gives a snapshot of a muti-period valuation as single-period one.

Purposes of this Paper

- The major concern in this paper are
  - the underlying asset is non-tradeable and hard to form a replicated portfolio
  - insufficient transaction data
  - provide the distribution in a general form that can be transformed into a normal distribution instead of distorting the underlying distribution as Wang transform.
- Try to provide another approach
  - pricing MLCCs in a general equilibrium setting
  - the risk neutral valuation relationship (RNVR) borrowed from Rubinstein (1976) and Brennan (1979) is still obtained.

Transformed Normal distribution

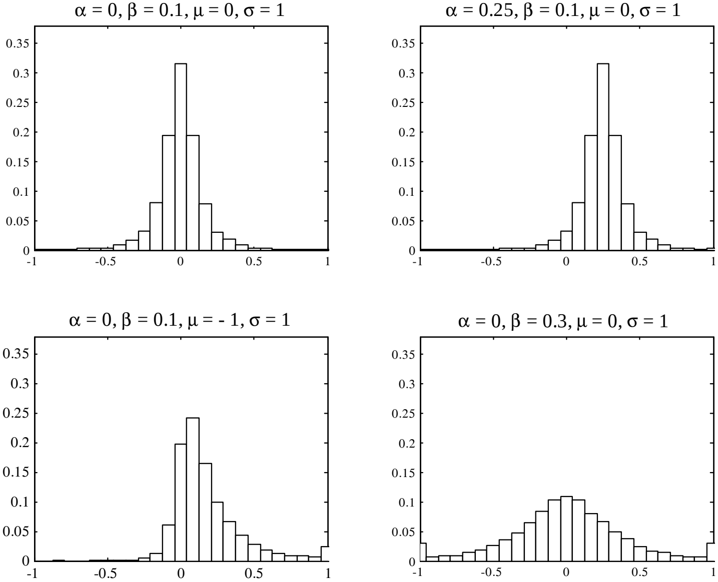
DEFINITION (Johnson 1949) The transformed normal distribution are defined by the transformation of random variable q such that:

$$f(\frac{q-\alpha}{\beta}) = x \backsim N(\mu, \sigma^2) \tag{6}$$

where α, β, μ and σ are parameters $(\beta, \sigma > 0)$ and f is a strictly monotonic differential function. $N(\mu, \sigma^2)$ is a normal distribution with mean μ and variance σ^2.
Example:

1. if $\alpha = 0$, $\beta = 1$ and f is $\log$ function, then q is a lognormal distribution
2. if f is inverse hyperbolic sine, i.e. $f(y) = \ln(y + \sqrt{y^2+1})$ then q is a S_U system

system S_U distributions



Equilibrium Pricing: Assumption 1

- The Model:

An non-satiated and risk-averse representative agent, maximize his expected utility:

$$\underset{C_o,\ y_j}{Max} \;\; U_0(C_0) + E^P \left\{ U \left[(W_0 - C_0)\, e^r + \sum_{j=1}^{n} y_j (P_{j1}(q) - P_{j0} e^r) \right] \right\}$$

where

- C_0: current consumption
- W_0: initial wealth
- $P_{j0}(q)$: the current price of security $j$written on underlying q.
- $P_{j1}(q)$: the payoff of the security $j$written on underlying $q$at the end-of-period.
- $y_j$is the demand for the securities.

Continued

- The equilibrium condition for the maximum:

$$P_{j0}(q) = e^{-rT} \frac{E^P\left[U'(W_1)P_{j1}(q)\right]}{E^P\left[U'(W_1)\right]} = e^{-rT} E^P\left[\phi(q)P_{j1}(q)\right] \tag{7}$$

where $\phi(q) = \frac{U'(W_1|q)}{E^P[U'(W_1)]}$ is the pricing kernel.

- We focus on one underlyings and one contingent claim written on it. The subscripts j and k can be suppressed:

$$P_0 = e^{-rT} E^P\left[\phi(q)P_1(q)\right], \tag{8}$$

the price of securities can be expressed as the expected value product by its relative conditional marginal utility and discounted at the riskfree rate.

Equilibrium Pricing: Assumption 2 and 3

- Camara (2003) assume the underlying assets q and wealth W have a joint distribution:

$$(f(W_1),\ f_1(q)) \sim \mathbf{N}(\mu_w, \mu, \sigma_w, \sigma, \rho) \tag{9}$$

  where h and h_1 are a strictly monotonic differentiable functions, $\mathbf{N}$ denotes the bivariate normal distribution with means μ_w and μ standard deviations σ_w and σ and correlation coefficient ρ.

- Assumptions 3: The marginal utility of the represent agent is:

$$U'(W_1) = \exp^{\beta f(W_1)} \tag{10}$$

  where β is constant and h is the same one as in equation (9).

Equilibrium Pricing: Results

- Camara (2003) showed that the equilibrium price is:

$$P = e^{-rT} E^{Q} \left[P_1(q) \right] \tag{11}$$

  where $E^{Q}[\cdot]$ is the expected value operator under the Q probability with respect to the risk-neutral transformed normal density.
- This risk-neutral transformed normal density has a shifted location parameter does not relate to preference parameters.
- The value of the contingent claim is the expected cashflow of $P_1(q)$ discounted by risk-free rate r.
- We use equation (11) to price MLCCs and discount their future payoff by the risk-free rate.

Model Features

- do not need transaction data
- the transformed normal distribution can brings high-order moments into the pricing formula better
  - for example, skewness or catastrophe risk
- this approach places more restrictive assumptions (underlying distribution, individual wealth, and preference) but get a simple pricing formula
  - No-arbitrage pricing does not need these restrictions

Specify the underlying distributions

- Specify underlying distributions to get MLCC pricing functions in closed-form solutions as Black-Scholes-type formula
  - standard lognormal distribution
    - the results of Rubinstein (1976) and Brennan (1979).
  - skewed lognormal distribution
  - S_U distribution

Su option model

- If W has a lognormal distribution, the mortality rate q_t has S_U system and representative agent has a power utility, i.e.,
  1. $f(W_1) = \ln(W_1)$,
  2. $f_1(q) = \sinh^{-1}(\frac{q-\alpha}{\beta}) = \ln(\frac{q-\alpha}{\beta} + \sqrt{1+(\frac{q-\alpha}{\beta})^2}) \backsim N(\mu, \sigma)$
  3. $\left(\ln(W_1), \sinh^{-1}(\frac{q-\alpha}{\beta})\right) \backsim \mathbf{N}(\mu_w, \mu, \sigma_w, \sigma, \rho)$
- With the terminal payoff $Max(q-K, 0)$, the option price at $t=0$ is:

$$P_0 = \frac{\beta}{2} e^{-rT + \mu^Q + \frac{1}{2}\sigma^2 T} \cdot \Phi(d_1) - \frac{\beta}{2} e^{-rT + \mu^Q + \frac{1}{2}\sigma^2 T} \cdot \Phi(d_2) + (\alpha - K) e^{-rT} \cdot \Phi(d_3) \quad (12)$$

where

$$\mu^Q = \sinh^{-1}\left(\frac{1}{\beta} e^{-\frac{1}{2}\sigma^2 T}(q_0 e^{rT} - \alpha)\right) \quad (13)$$

$$d_1 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma\sqrt{T}} + \sigma\sqrt{T}$$

$$d_2 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$

$$d_3 = \frac{-\sinh^{-1}(\frac{K-\alpha}{\beta}) + \mu^Q}{\sigma\sqrt{T}}$$

and $\Phi(\cdot)$ is cumulated standard normal distribution.

- Proof: see Appendix A.

Parameter Estimation for Su distribution

- quantile-based estimation method of Slifker and Shapiro (1980)
- choose any value $z > 0$ from a standard normal random variable (for example, choose $z = 1$). Then the four points $\pm z$ and $\pm 3z$ determine the corresponding value of the raw data. They are q_{-3z}, q_{-z}, q_z and q_{3z}. Let

$$m = q_{3z} - q_z$$
$$n = q_{-z} - q_{-3z}$$
$$p = q_z - q_{-z}.$$

Parameter Estimation for Su distribution

- If the data passes the criteria of Su distribution, $mn/p^2 > 1$, then the estimates for the parameters are:

$$\alpha = \frac{x_z + x_{-z}}{2} + \frac{n-m}{2(\frac{m}{p} + \frac{n}{p} - 2)};$$

$$\beta = \frac{2p\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}{\left(\frac{m}{p} + \frac{n}{p} - 2\right)\left(\frac{m}{p} + \frac{n}{p} + 2\right)^{1/2}}; \quad (\beta > 0)$$

$$\mu = \sinh^{-1}\left[\frac{\frac{m}{p} - \frac{n}{p}}{2\left(\frac{m}{p}\frac{n}{p} - 1\right)^{1/2}}\right];$$

$$\sigma = \frac{\cosh^{-1}\left[\frac{1}{2}\left(\frac{m}{p} + \frac{n}{p}\right)\right]}{2z}. \quad (\sigma > 0)$$

Simulation data

- use stochastic mortality processes to generate the terminal distribution of q.
  - catastrophe model of Lin and Cox (2008)
  - Lee-Carter with jumps model (Chen and Cox, 2009),
- Mortality data are from the National Center for Health Statistics (NCHS)
  - NCHS reports the United States age-adjusted death rate per 100,000 standard million population for selected causes of death. Our data is form 1900 to 1998.
  - use this data to generate the mortality distribution from 2003 to 2005,
  - estimate the parameters of the distribution according to the transformed normal distribution.

Parameter for Su Distribution

- The estimated parameters of mortality data from 2003 to 2005 and $q_{\max}$ are shown in the Table 1.

Table 1 The parameters of mortality data assuming following Su distribution

		q_{2003}	q_{2004}	q_{2005}	$q_{\max}$
Basic statistics	mean	0.0079608	0.0072285	0.0065657	0.0079698
	standard deviation	0.0002845	0.0004107	0.00051437	0.0002957
	skewness	1.4226	0.52041	0.36981	1.5167
	kurtosis	10.658	4.6036	3.5588	10.473
Transformed parameters	α	0.0076907	0.0067493	0.0055682	0.0076918
	β	0.0002611	0.0007357	0.0016061	0.0002638
	μ	0.73272	0.55769	0.56733	0.73965
	σ	0.68841	0.44484	0.26635	0.68807

Options and Mortality Bond Prices

Table 2 The prices of calls and mortality bond under Su distribution

	Without Approximation			Approximation
Present value of	t=2003	t=2004	t=2005	
call 1 = $Max(q_t-K_1, 0)$	0.020277	0.019170	0.0079474	0.020199
call 2 = $Max(q_t-K_2, 0)$	0.005293	0.002997	0.0004304	0.005264
L_t	8.5544	9.2334	4.2915	8.5267
Bond price			977.92	991.47
Par spread			74 bps	29 bps

Conclusions and Discussion

We provide an equilibrium pricing approach to value MLCCs and apply it on the Swiss Re mortality bond.

1. an convenient and explicit valuation formulation is obtained.
2. do not require market transaction data and replicated portfolio assumptions
3. the valuation is preference-free and the payoff could be discounted at risk-free rate, too.
4. we assume a more general distribution that can be transformed into normal distribution and enables the inclusion of high-order moments, when the mortality jump is important to the valuation.

Conclusions and Discussion

Two preliminary conclusions in our results

- The approximation method would undervalue the mortality bond price.
    - the approximation method increases the mean, but decreases the variance of the mortality rate.
- Whether using the approximation method or not, both spreads are smaller than the level offered by Swiss Re.
    - the default risk or loading fees are not considered.
    - if the friction costs is small or could be omitted, it may be that Swiss Re over-compensated mortality bond investors.

Thanks for listening !