Real World Pricing of Long Term Contracts

Eckhard Platen

School of Finance and Economics and Department of Mathematical Sciences

University of Technology, Sydney

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Introduction

- **variable annuities**
  fund-linked
  tax-deferred
  guarantees

- **guaranteed minimum death benefits** (GMDBs)

  roll-ups:
  original investment accrued at a pre-defined interest rate

- GMDB put, floating put and/or look-back put option

  long maturities of contracts

  standard option pricing theory ?

  no obvious best choice for price:

  IFRS Phase II

  CFO-Forum (2008)

  CRO-Forum (2008)

  Solvency II

- **pricing and hedging of GMDBs**

  benchmark approach in

  Pl. (2002, 2004), Pl. & Heath (2006)

  Bühlmann & Pl. (2003)

  best performing portfolio as benchmark

  does **not** require
  existence of an equivalent risk neutral probability measure

  exploits real trends

  may provide significantly lower prices

Financial Model

- **underlying risky security** (unit)

$$dS_t = (\mu_t - \gamma) S_t dt + \sigma_t S_t \, dW_t$$

$\gamma \geq 0$ management fee rates

W_t - Brownian motion

- **savings account**

$$dB_t = r_t B_t \, dt$$

- **market price of risk**

$$\theta_t = \frac{\mu_t - \gamma - r_t}{\sigma_t}$$

- **risky asset**

$$\frac{dS_t}{S_t} = r_t\, dt + \sigma_t \theta_t\, dt + \sigma_t\, dW_t$$

- **instantaneous portfolio return**

$$\frac{dV_t}{V_t} = \pi_t^0 \frac{dB_t}{B_t} + \pi_t^1 \frac{dS_t}{S_t}$$

$$= r_t\,dt + \pi_t^1 \sigma_t(\theta_t\,dt + dW_t)$$

- **fractions**

$$\pi_t^0 = \delta_t^0 \frac{B_t}{V_t}, \qquad \pi_t^1 = \delta_t^1 \frac{S_t}{V_t}$$

$$\pi_t^0 + \pi_t^1 = 1$$

- **numeraire portfolio** (NP) as benchmark

  Long (1990)
  V^* is best performing in several ways

- is growth optimal portfolio

  Kelly (1956)

$$V^* = \max E(\log V_T)$$

$$\pi_t^{1*} = \frac{\mu_t - \gamma - r_t}{\sigma_t^2} = \frac{\theta_t}{\sigma_t}$$

$$\frac{dV_t^*}{V_t^*} = r_t\,dt + \theta_t(\theta_t\,dt + dW_t)$$

  Merton (1992)

- **NP best performing portfolio**

$$\limsup_{T\to\infty} \frac{1}{T} \log\left(\frac{V_T^*}{V_0^*}\right) \geq \limsup_{T\to\infty} \frac{1}{T} \log\left(\frac{V_T}{V_0}\right)$$

global well diversified index $\approx$ NP

Pl. (2005)

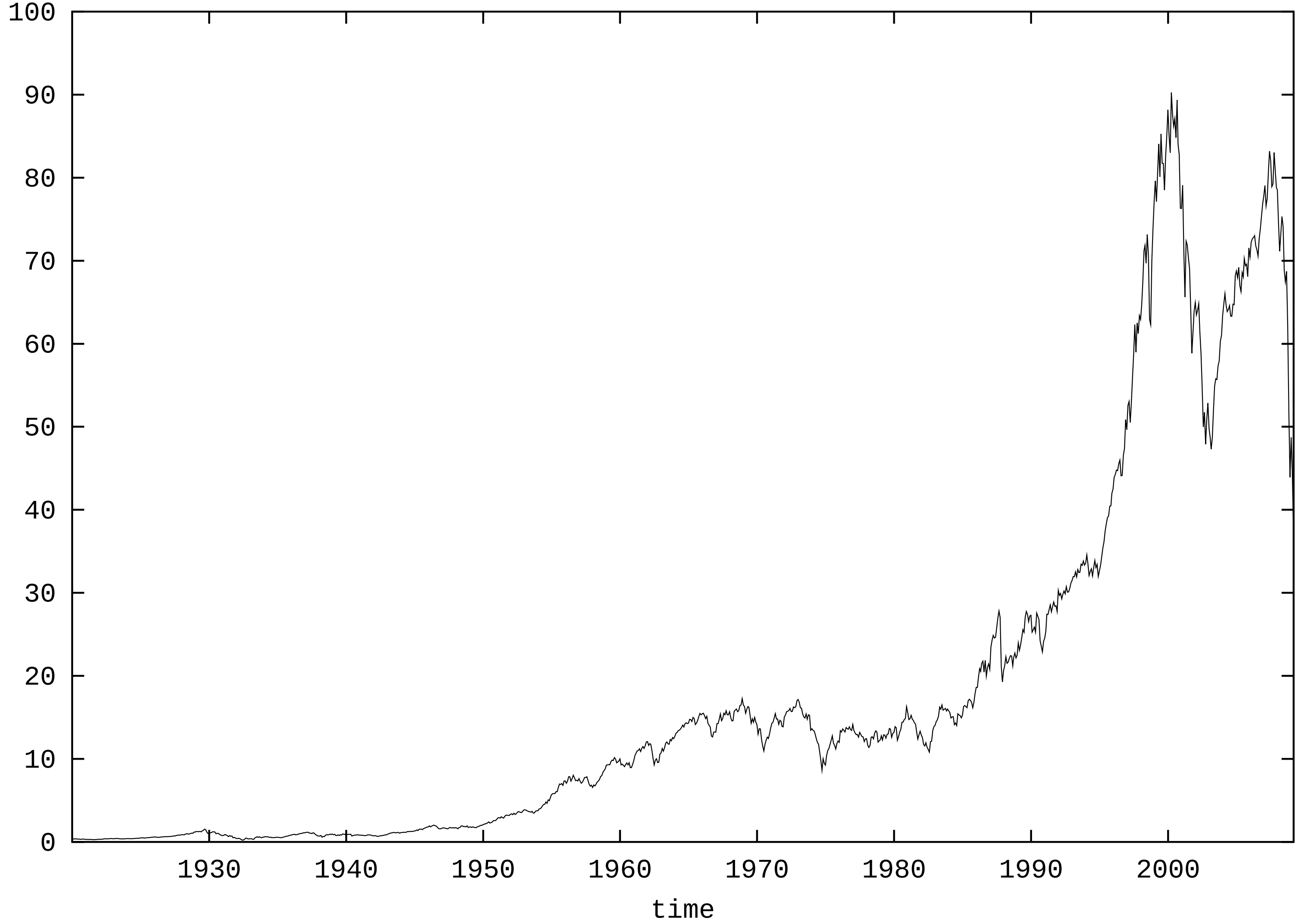

Figure 1: Discounted S&P500 total return index.

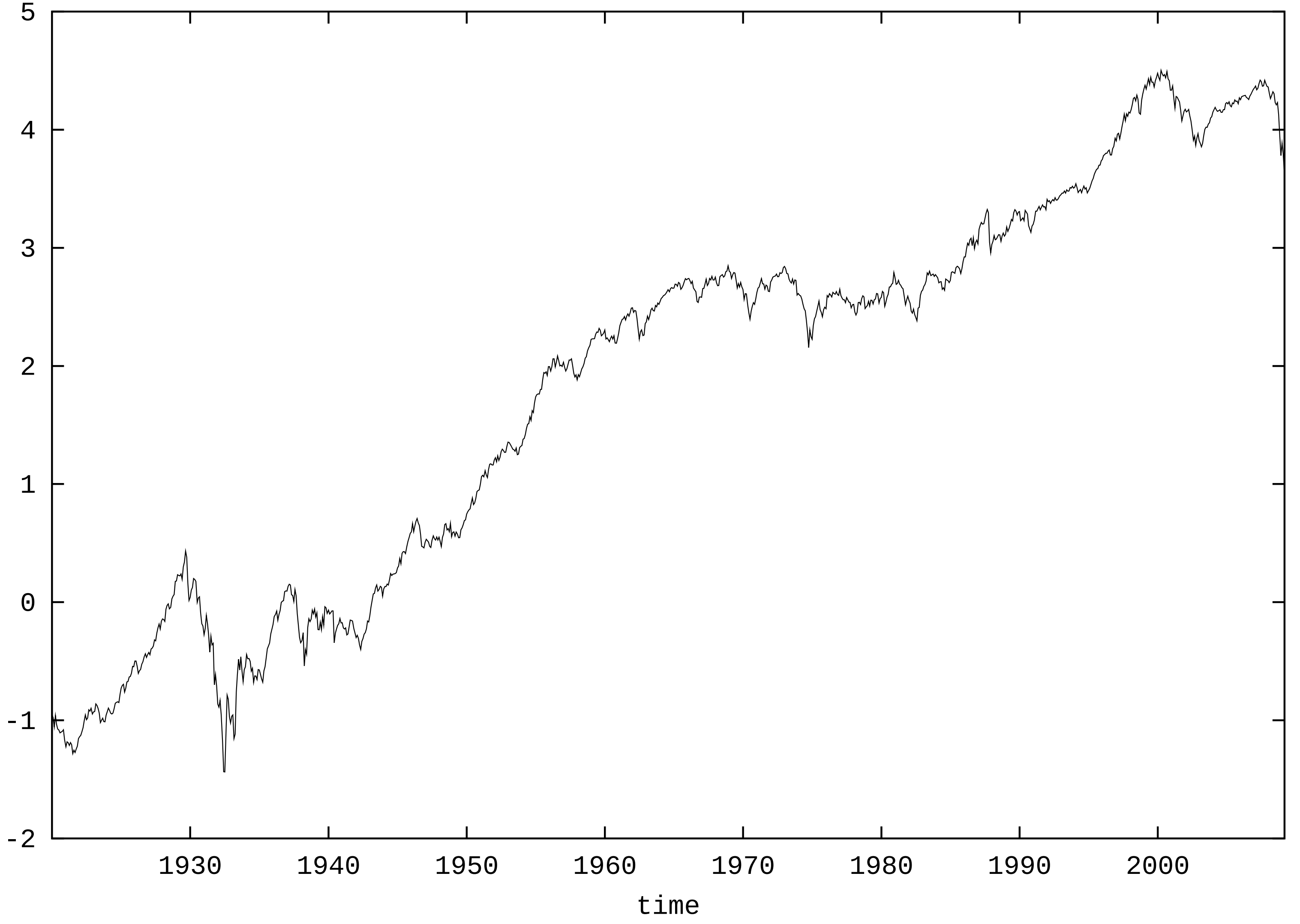

Figure 2: Logarithm of discounted S&P500.

- **benchmarked price**

  $\hat{U}_t = \frac{U_t}{V_t^*}$

  $\hat{U}$ nonnegative $\Longrightarrow$

$$\hat{U}_t \geq E_t\left(\hat{U}_s\right)$$

  $t \leq s$

  supermartingales (no upward trend)

  Pl. (2002)

  no strong arbitrage

- benchmarked securities that form martingales are called fair.

$$\hat{U}_t = E_t \left(\hat{U}_s \right)$$

$t \leq s$

- **Law of the Minimal Price**

  Pl. (2008)

  Fair prices are minimal prices

  V nonnegative fair portfolio
  V' nonnegative portfolio

$$V_T = V'_T$$

supermartingale property
$\Longrightarrow$

$$V_t \leq V'_t$$

$t \in [0, T]$,

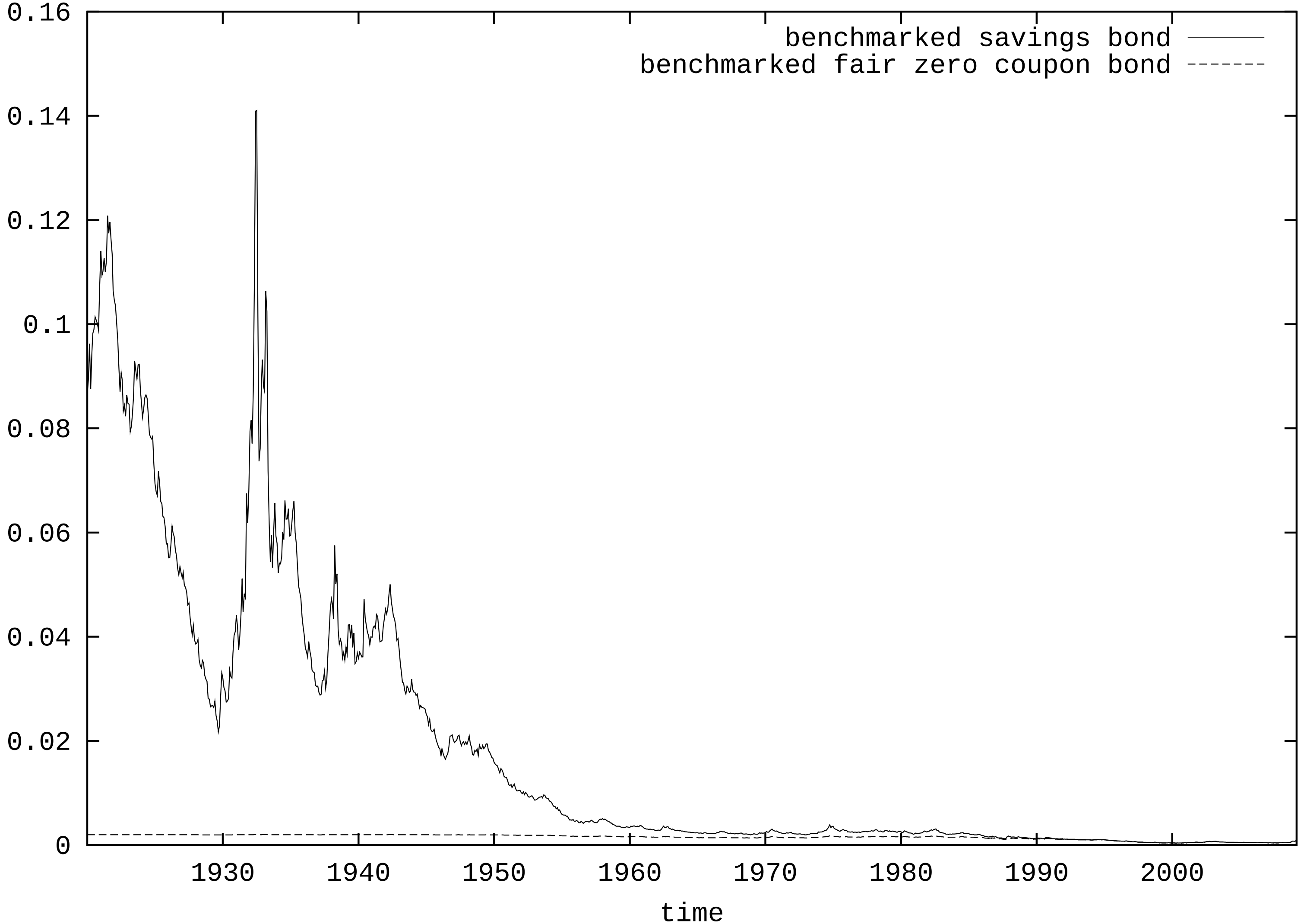

Figure 3: Benchmarked savings bond and benchmarked fair zero coupon bond.

- **real world pricing formula**

  for

$$E_t\left(\frac{H_T}{V_T^*}\right) < \infty$$

$$U_H(t) = V_t^*\, E_t\left(\frac{H_T}{V_T^*}\right)$$

$t \in [0, T]$

no risk neutral probability needs to exist

- **actuarial pricing formula**

  when H_T is independent of V_T^*

  $\Longrightarrow$

$$U_H(t) = P(t,T)\, E_t(H_T)$$

  zero coupon bond

$$P(t,T) = V_t^*\, E_t\left(\frac{1}{V_T^*}\right)$$

- **standard risk neutral pricing**

Ross (1976), Harrison & Pliska (1983)

complete market

candidate Radon-Nikodym derivative

$$\Lambda_t = \left.\frac{dQ}{dP}\right|_{\mathcal{A}_t} = \frac{B_t V_0^*}{B_0 V_t^*}$$

supermartingale $\Longrightarrow$

$$1 = \Lambda_0 \geq E_0\left(\Lambda_T\right)$$

$\Longrightarrow$

$$
\begin{aligned}
U_H(0) &= E\left(\frac{V_0^*}{V_T^*}\,H_T\right) \\
&= E\left(\Lambda_T\,\frac{B_0}{B_T}\,H_T\right) \leq \frac{E\left(\Lambda_T\,\frac{B_0}{B_T}\,H_T\right)}{E(\Lambda_T)}
\end{aligned}
$$

Only in the special case when Λ_T martingale

$\Longrightarrow$ risk neutral pricing formula:

$$U_H(0) = E_Q\left(\frac{B_0}{B_T}\,H_T\right)$$

Ignores any real trend !

- **Trends ignored also when using:**

  stochastic discount factor, Cochrane (2001);

  deflator, Duffie (2001);

  pricing kernel, Constantinides (1992);

  state price density, Ingersoll (1987);

  NP as Long (1990)

- **Example zero coupon bond**

  benchmarked fair zero coupon

$$\hat{P}(t,T) = \frac{P(t,T)}{V_t^*} = E_t\left(\frac{1}{V_T^*}\right)$$

martingale (no trend)

for r_t - deterministic

$$P(t,T) = V_t^* \, E_t\left(\frac{1}{V_T^*}\right) = \exp\left\{-\int_t^T r_s \, ds\right\} E_t\left(\frac{\bar{V}_t^*}{\bar{V}_T^*}\right),$$

$\bar{V}_t^* = \frac{V_t^*}{B_t}$ - discounted NP

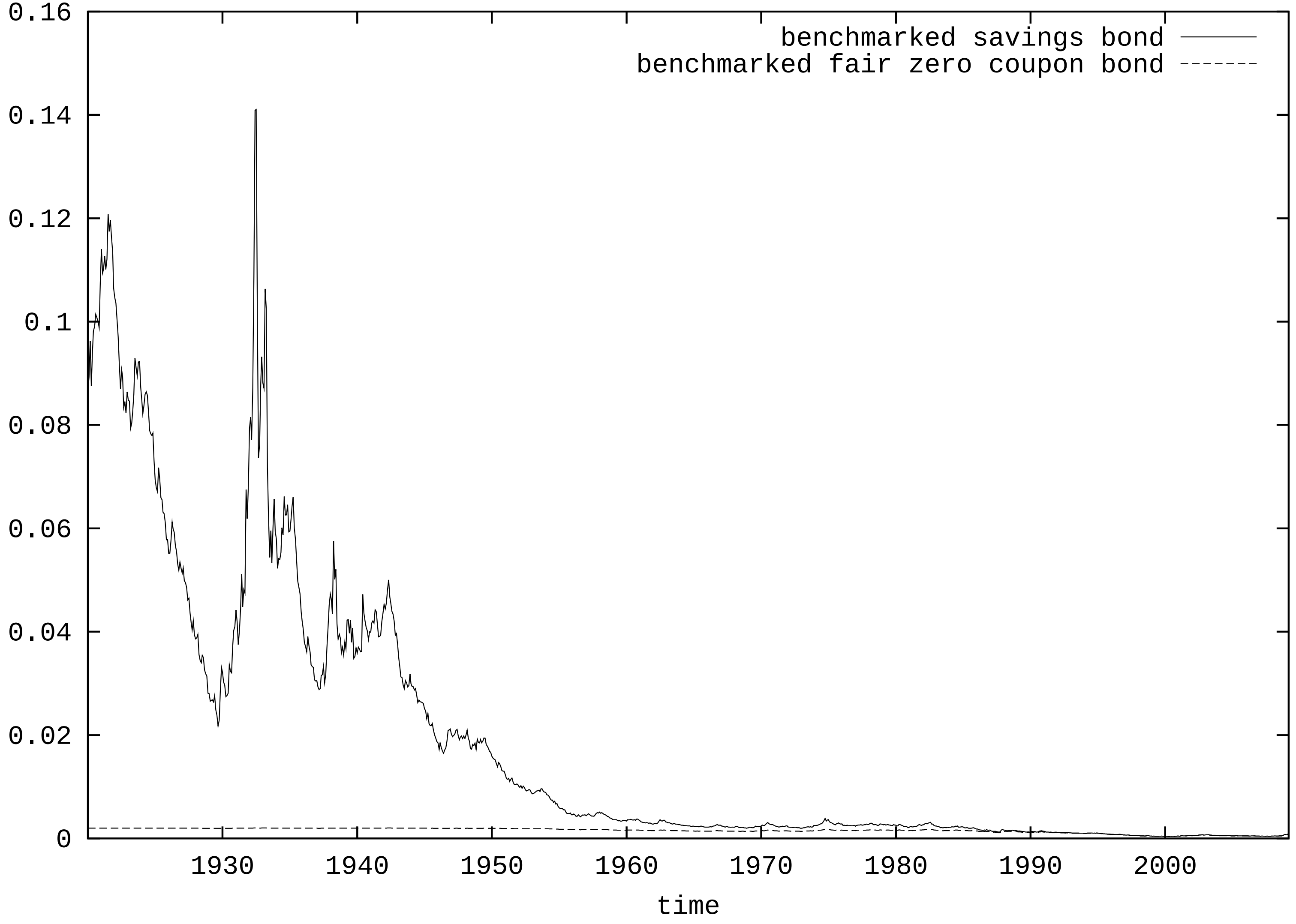

Figure 4: Benchmarked savings bond and benchmarked fair zero coupon bond.

$\Longrightarrow$ downward trend reflects equity premium

$\Longrightarrow$ $\boldsymbol{\Lambda}$ strict supermartingale (downward trend), then

$$P(t,T) < \exp\left\{-\int_t^T r_s\,ds\right\} = \frac{B_t}{B_T}$$

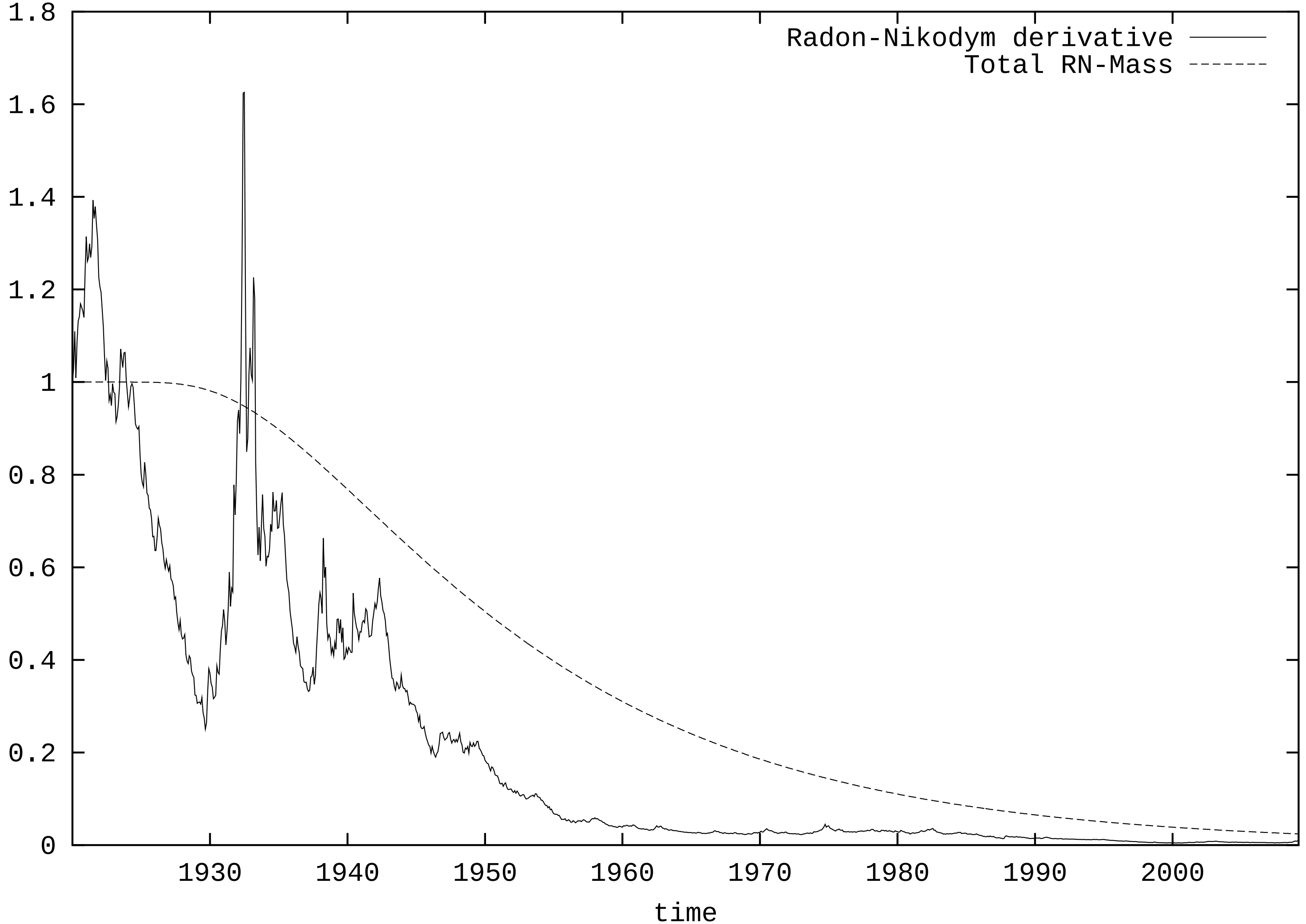

Figure 5: Radon-Nikodym derivative and total mass of putative risk neutral measure.

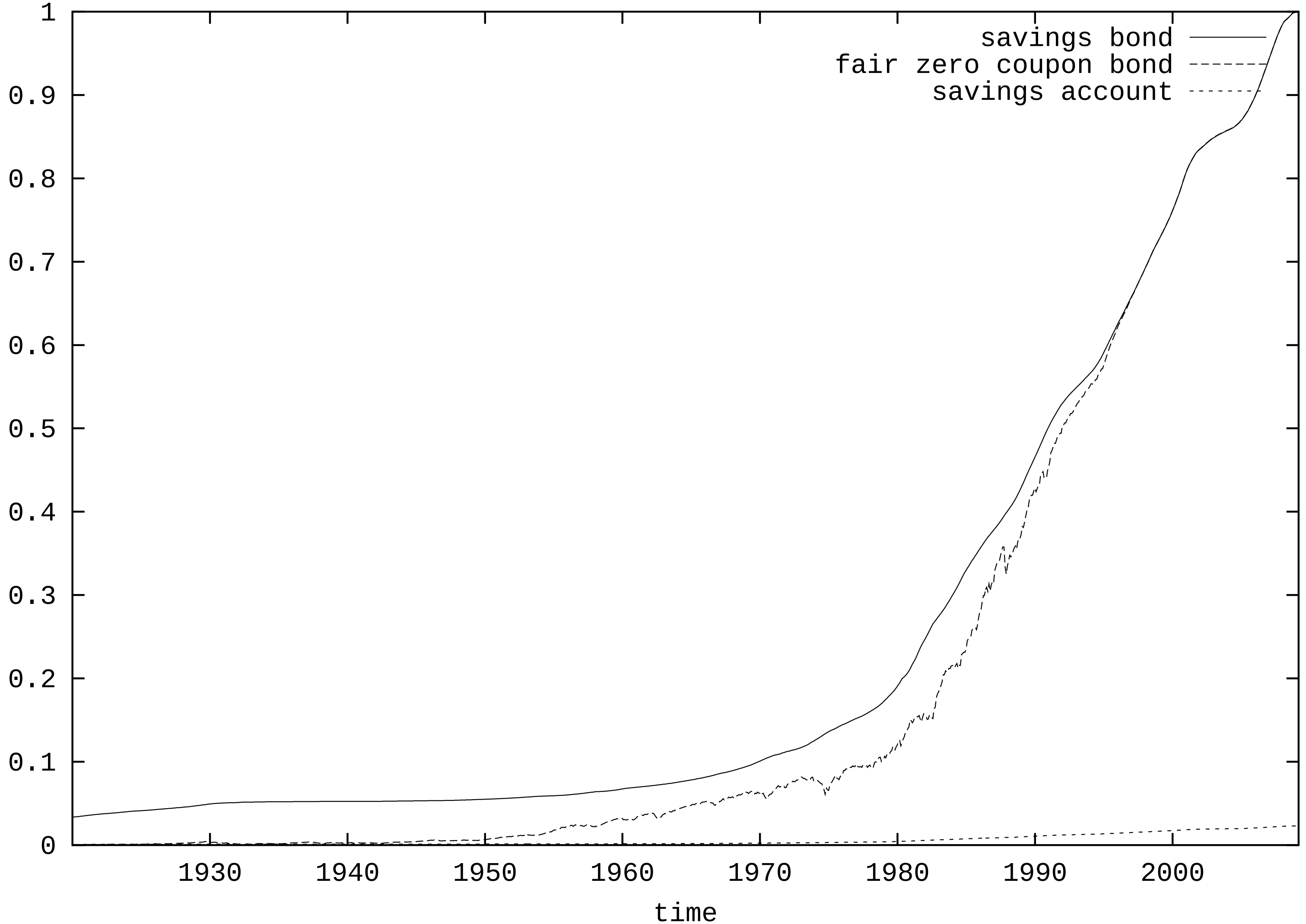

Figure 6: Savings bond, fair zero coupon bond and savings account.

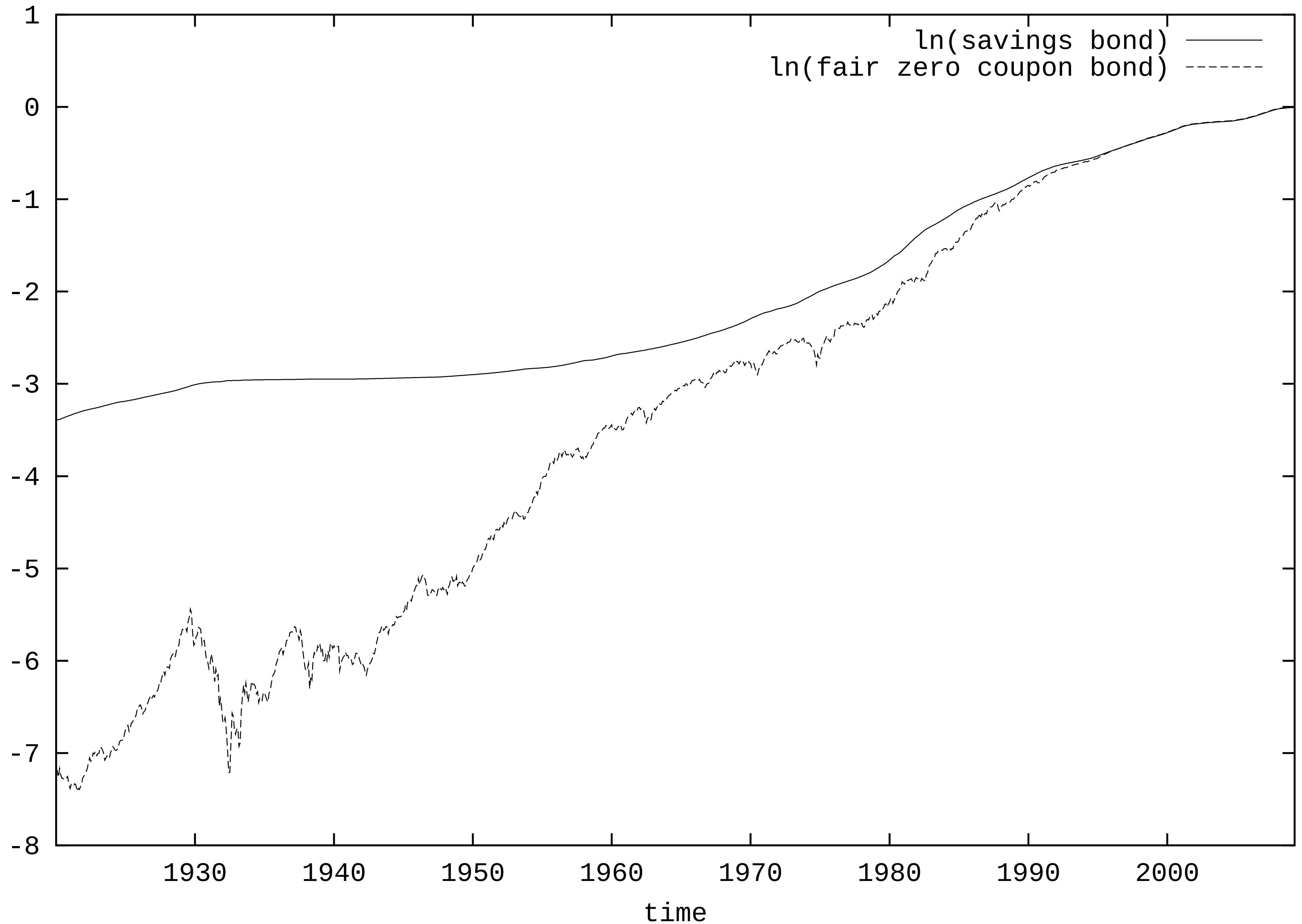

Figure 7: Logarithms of savings bond and fair zero coupon bond.

- discounted NP

$$\bar{V}_t^* = \frac{V_t^*}{B_t}$$

$$\begin{aligned} d\bar{V}_t^* &= \bar{V}_t^* \, \theta_t \, (\theta_t \, dt + dW_t) \\ &= \alpha_t \, dt + \sqrt{\alpha_t \, \bar{V}_t^*} \, dW_t \end{aligned}$$

- discounted NP drift

$$\alpha_t := \bar{V}_t^* \theta_t^2$$

$\Longrightarrow$ volatility

$$\theta_t = \sqrt{\frac{\alpha_t}{\bar{V}_t^*}}$$

reflects leverage effect

- **minimal market model**

  Pl. (2001, 2002)

  MMM

  discounted NP drift

  assume

$$\alpha_t = \alpha_0 \exp\{\eta t\}$$

  $\alpha_0 > 0$

  net growth rate $\eta > 0$

  $\Longrightarrow$ MMM

- **normalized NP**

$$Y_t = \frac{\bar{V}_t^*}{\alpha_t}$$

$$dY_t = (1 - \eta Y_t)\,dt + \sqrt{Y_t}\,dW_t$$

square root process of dimension four

Only one parameter !

- **volatility of NP**

$$\theta_t = \frac{1}{\sqrt{Y_t}} = \sqrt{\frac{\alpha_t}{\bar{V}_t^*}}$$

leverage effect

- **Zero coupon bond under the MMM**

  r_t - deterministic

$$P(t,T) = \exp\left\{-\int_t^T r_s\,ds\right\}\left(1 - \exp\left\{-\frac{\bar{V}_t^*}{2(\varphi(T)-\varphi(t))}\right\}\right)$$

Explicit formula !

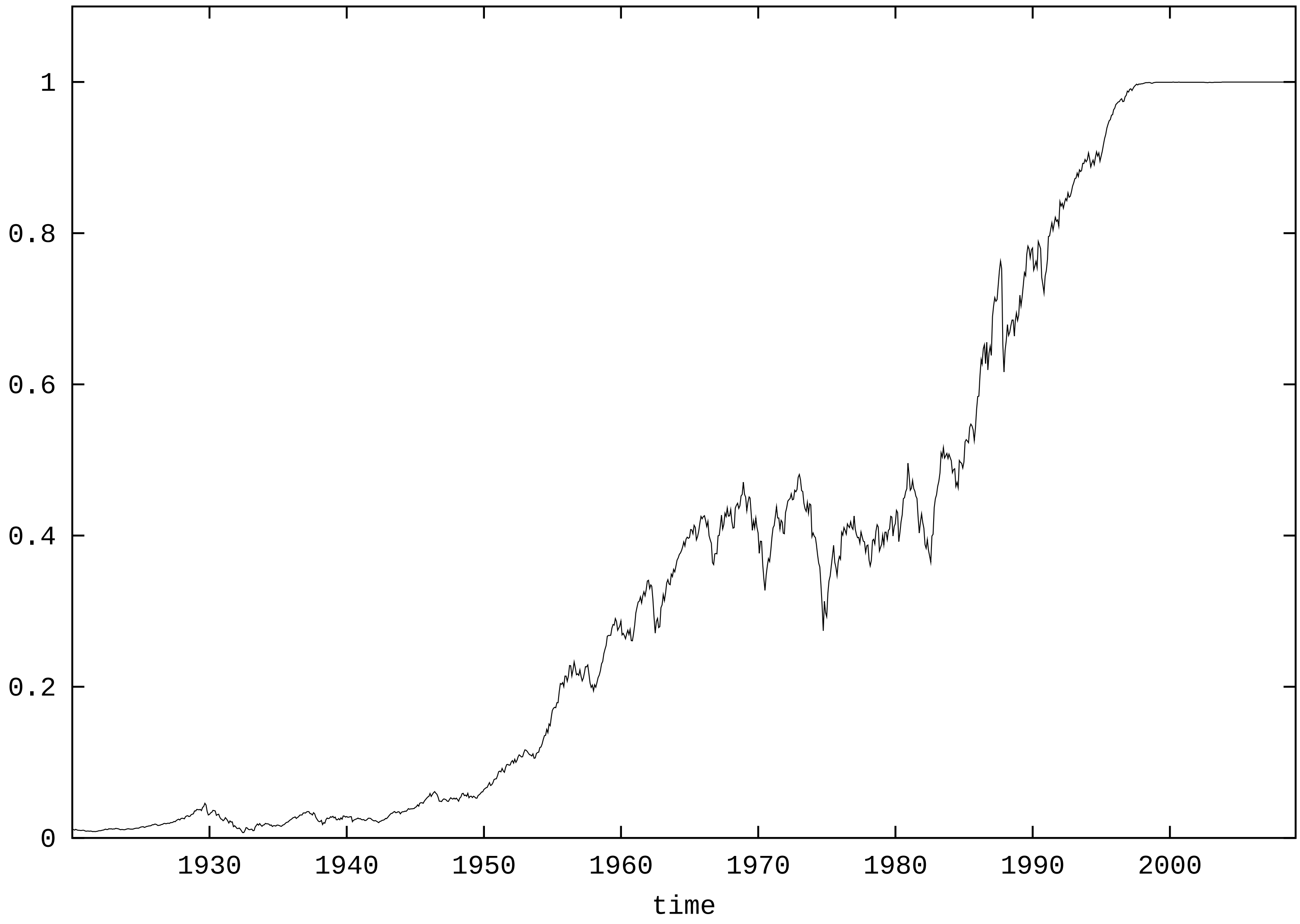

Figure 8: Fraction invested in the savings account.

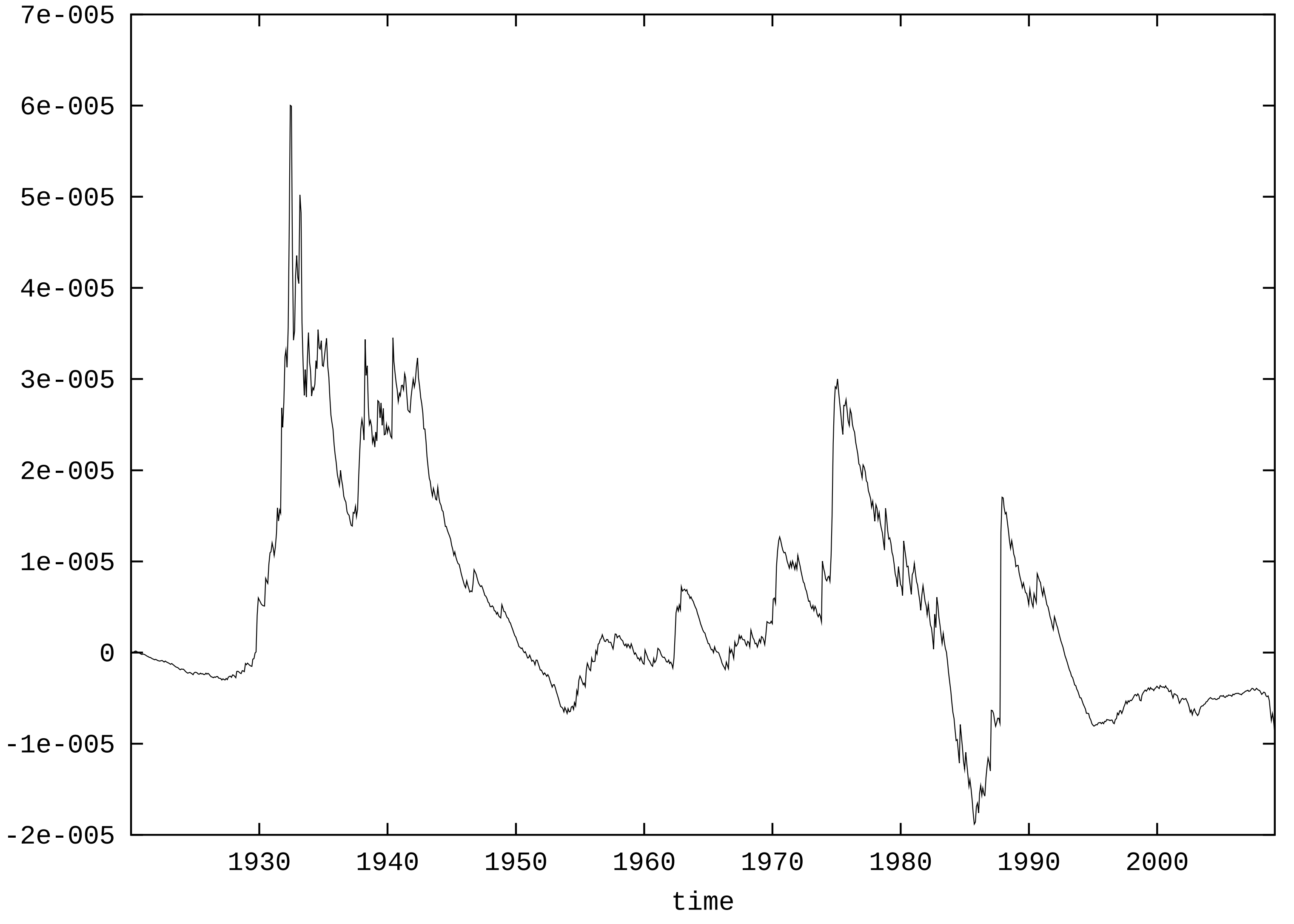

Figure 9: Benchmarked profit and loss.

- **European put option under the MMM**

  Hulley, Miller & Pl. (2005)

$$p(t, V_t^*, T, K, r) = V_t^* E_t\left(\frac{(K - V_\tau^*)^+}{V_\tau^*}\right)$$

$$\begin{aligned} p(t, V_t^*, T, K, r) &= -V_t^* \chi^2(d_1; 4, l_2) \\ &\quad + K e^{-r(T-t)}\left(\chi^2(d_1; 0, l_2) - \exp\{-l_2/2\}\right) \end{aligned}$$

with

$$\boldsymbol{d_1 = \frac{4\eta K \exp\{-r(T-t)\}}{B_t \alpha_t (\exp\{\eta(T-t)\} - 1)}}$$

and

$$\boldsymbol{l_2 = \frac{4\eta V_t^*}{B_t \alpha_t (\exp\{\eta(T-t)\} - 1)}}$$

$\chi^2(x; n, l)$ non-central chi-square distribution function

$n \geq 0$ degrees of freedom

non-centrality parameter $l > 0$

$$\chi^2(x; n, l) = \sum_{k=0}^{\infty} \frac{\exp\left\{-\frac{l}{2}\right\}\left(\frac{l}{2}\right)^k}{k!}\left(1 - \frac{\Gamma\left(\frac{x}{2}; \frac{n+2k}{2}\right)}{\Gamma\left(\frac{n+2k}{2}\right)}\right)$$

- **putative risk neutral price**

$$\tilde{p}(t, V_t^*, T, K, r) = p(t, V_t^*, T, K, r) + K e^{-r(T-t)} \exp\left\{ -\frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))} \right\}$$

risk neutral is overpricing

for $\bar{V}_t^* \to 0 \quad \Longrightarrow \quad \tilde{p}(t, V_t^*, T, K, r) \to 0$

and

$$\tilde{p}(t, V_t^*, T, K, r) \to K\, e^{-r(T-t)} > 0$$

risk neutral ignores trends

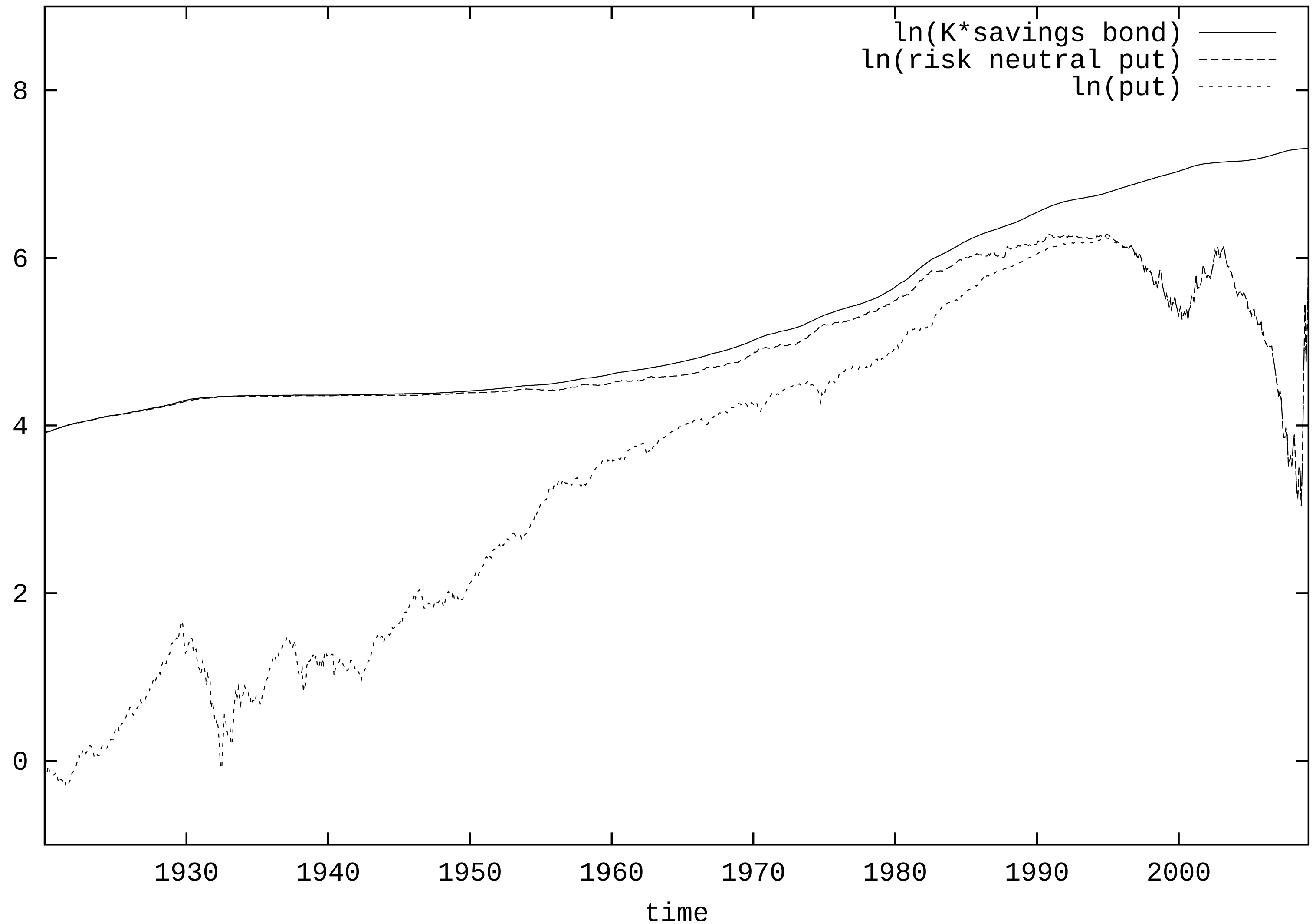

Figure 10: Logarithms of savings bond times $\boldsymbol{K}$, risk neutral put and fair put.

Guaranteed Minimum Death Benefit (GMDB)

- payout to the policyholder

$$\max(e^{g\tau}V_0, V_\tau)$$

time of death τ

$g \geq 0$ is the guaranteed instantaneous growth rate

V_0 is the initial account value

V_τ is the unit value of the policyholder's account at time of death τ

- insurance charges $\xi \geq 0$

$\Longrightarrow$ policyholder's unit value

$$V_t = e^{-\xi t} V_t^*$$

$\Longrightarrow$

$$\begin{aligned}\max(e^{g\tau} V_0, V_\tau) &= \max(e^{g\tau} V_0, e^{-\xi\tau} V_\tau^*)\\ &= e^{-\xi\tau} \max(e^{(g+\xi)\tau} V_0, V_\tau^*)\end{aligned}$$

insurance company invests the entire fund value V in the NP V^*

$\Longrightarrow$ payoff

$$H_T = GMDB_T = e^{-\xi T}\left[(e^{(g+\xi)T}V_0^* - V_T^*)^+ + V_T^*\right]$$

fair value $GMDB_0$ of the total claim

real world pricing formula

$$
\begin{aligned}
GMDB_0 &= V_0^* \, E\left(\frac{GMDB_T}{V_T^*}\right) \\
&= e^{-\xi T}\left(p(0, V_0^*, T, e^{(g+\xi)T} V_0^*, r) + V_0^*\right)
\end{aligned}
$$

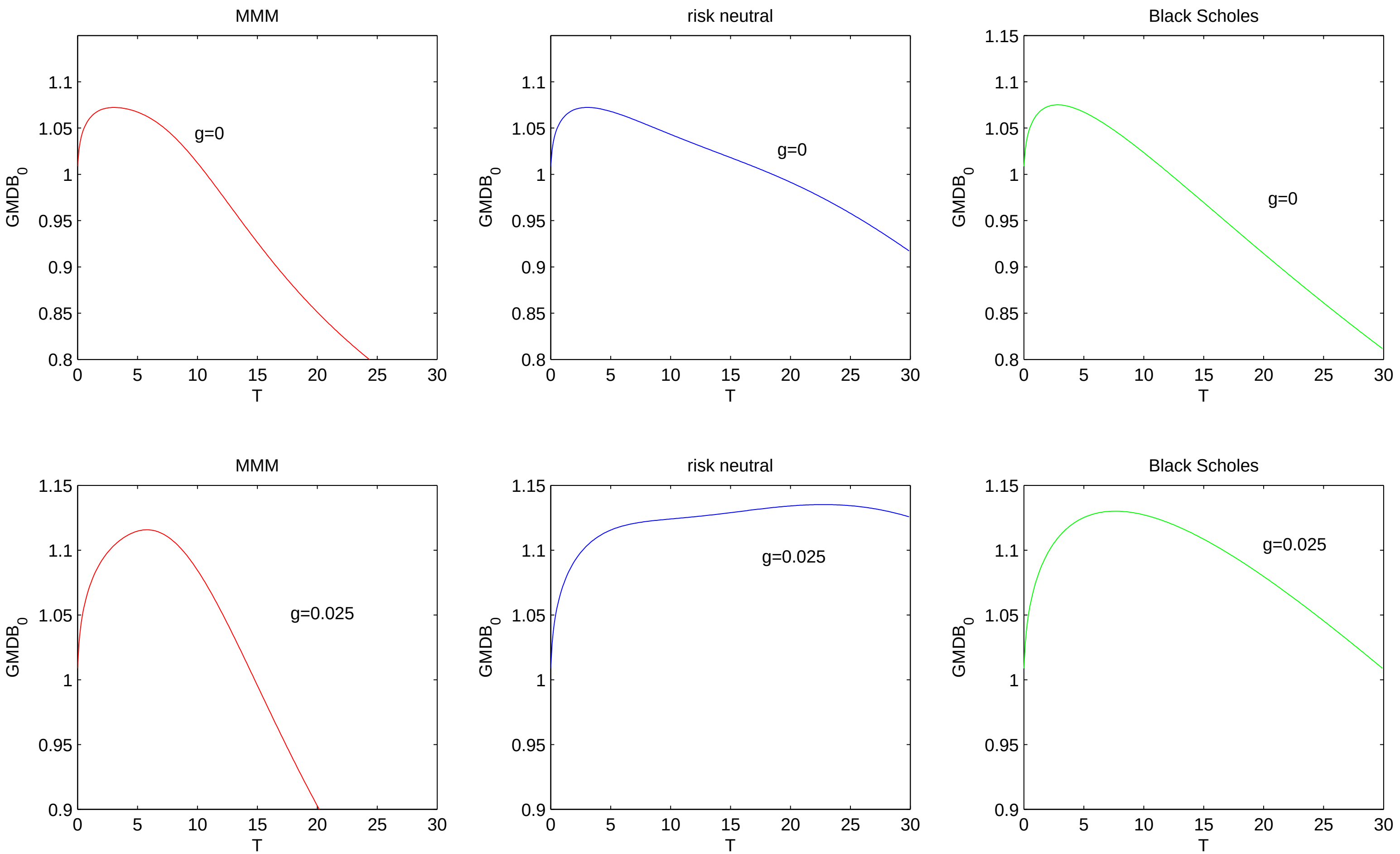

Figure 11: Present value of the GMDB under the real world pricing formula (left), the risk neutral pricing formula (middle) and the Black Scholes formula (right) for $\eta = 0.05$, $\alpha_0 = 0.05$, $r = 0.05$, $\xi = 0.01$ and $Y_0 = 20$.

- **lifetime τ is stochastic**

$$GMDB_0 = V_0^* \, E\left(E\left(\frac{GMDB_\tau}{V_\tau^*} \,\middle|\, \mathcal{F}_\tau \right)\right)$$

$$GMDB_0 = \int_0^T \left(p(0, V_0^*, t, e^{(g+\xi)t} V_0^*, r) + V_0^* \right) e^{-\xi t} f_\tau(t) \, dt$$

$f_\tau(\cdot)$ - mortality density

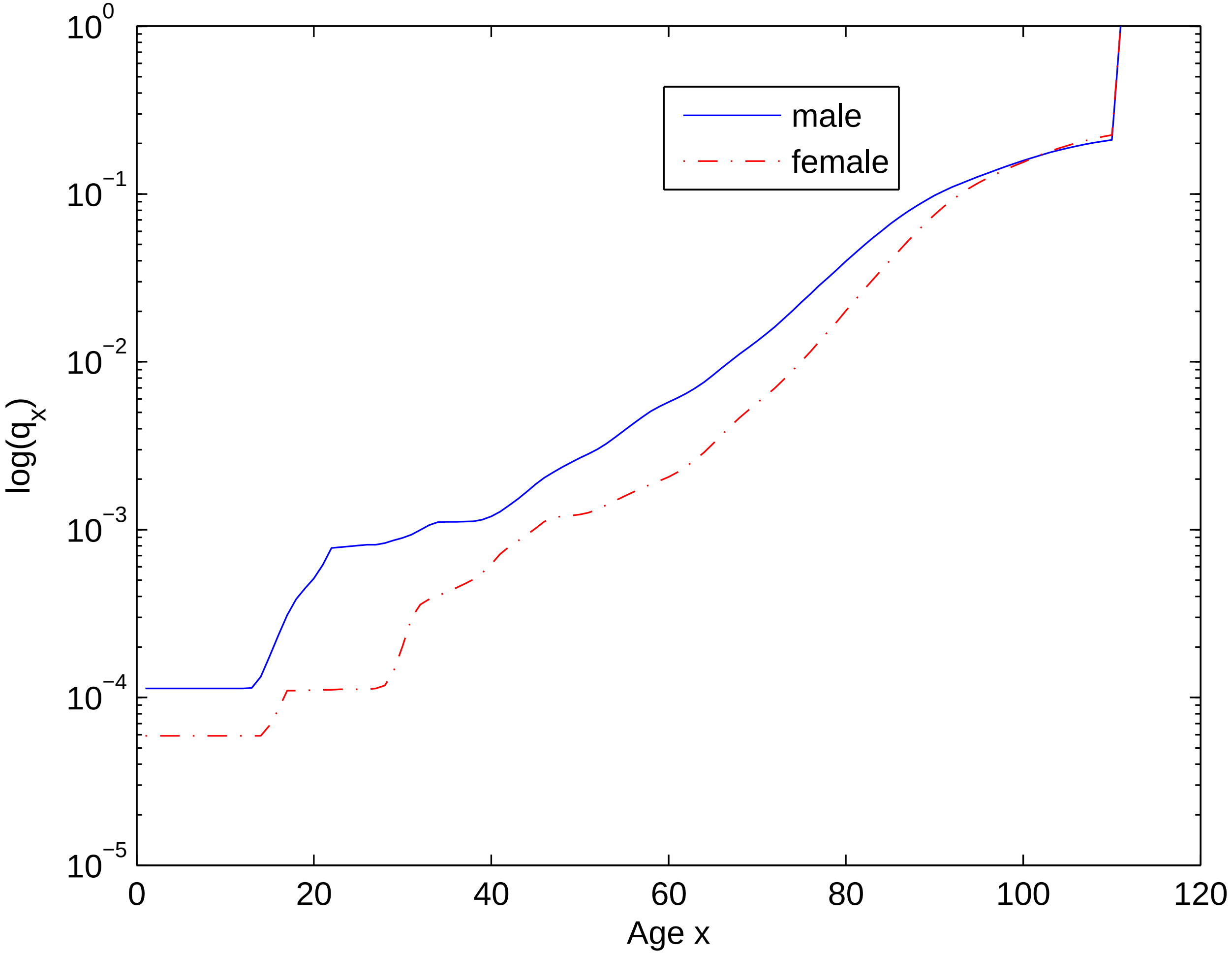

Figure 12: German mortality data

rational investors will lapse when embedded put-option out of the money

GMDBs have stochastic maturity

Titanic options, Milevsky & Posner (2001)

- **roll-up GMDB payoff**

$$H_t = \begin{cases} (1-\beta_t)V_t^*, & \text{if lapsed at time } t, \\ \max(e^{g\tau}V_0, V_\tau^*), & \text{if death occurs at time } t=\tau \end{cases}$$

surrender charge β_t

$$\beta_t = \begin{cases} (8-\lceil t \rceil)\%, & t \le 7, \\ 0, & t > 7 \end{cases}$$

no credit risk

no accumulation phase

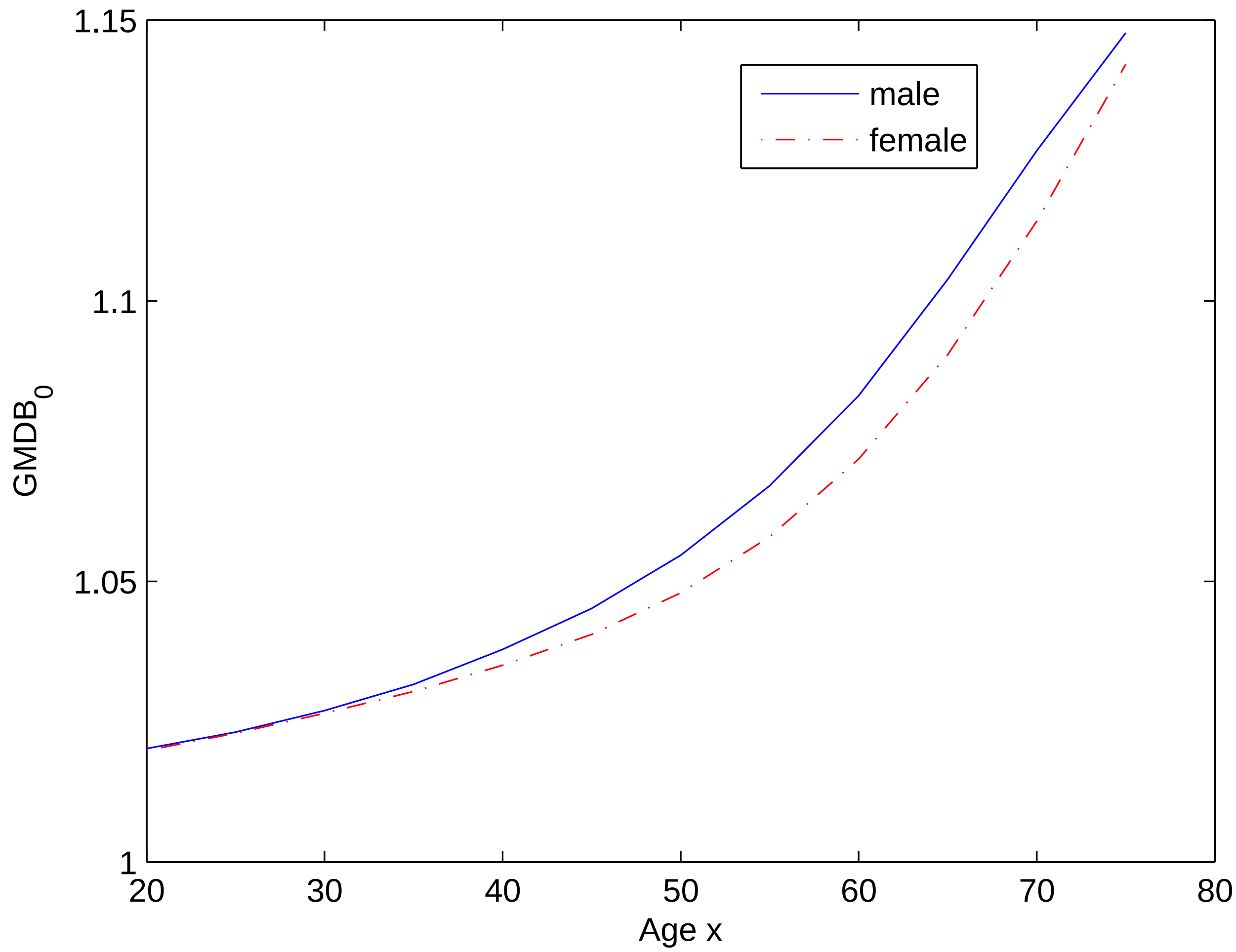

Figure 13: Value of the GMDB under the MMM for male and female policyholders aged x, assuming an irrational lapsation of $l = 1\%$.

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