Real World Pricing of Long Term Contracts

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Introduction

- variable annuities
  - fund-linked
  - tax-deferred
  - guarantees

- guaranteed minimum death benefits (GMDBs)
  - roll-ups:
    - original investment accrued at a pre-defined interest rate
GMDB put, floating put and/or look-back put option

long maturities of contracts

standard option pricing theory?

no obvious best choice for price:

IFRS Phase II
CFO-Forum (2008)
CRO-Forum (2008)
Solvency II
• pricing and hedging of GMDBs

benchmark approach in


best performing portfolio as benchmark
does not require
existence of an equivalent risk neutral probability measure
exploits real trends
may provide significantly lower prices
Financial Model

- **underlying risky security** (unit)

\[ dS_t = (\mu_t - \gamma)S_t \, dt + \sigma_t S_t \, dW_t \]

\[ \gamma \geq 0 \text{ management fee rates} \]

\( W_t \) - Brownian motion

- **savings account**

\[ dB_t = r_t B_t \, dt \]
• market price of risk

\[ \theta_t = \frac{\mu_t - \gamma - r_t}{\sigma_t} \]

• risky asset

\[ \frac{dS_t}{S_t} = r_t \, dt + \sigma_t \theta_t \, dt + \sigma_t \, dW_t \]
• instantaneous portfolio return

\[
\frac{dV_t}{V_t} = \pi_t^0 \frac{dB_t}{B_t} + \pi_t^1 \frac{dS_t}{S_t}
\]

\[
= r_t \, dt + \pi_t^1 \sigma_t (\theta_t \, dt + dW_t)
\]

• fractions

\[
\pi_t^0 = \delta_t^0 \frac{B_t}{V_t}, \quad \pi_t^1 = \delta_t^1 \frac{S_t}{V_t}
\]

\[
\pi_t^0 + \pi_t^1 = 1
\]
• **numeraire portfolio** (NP) as benchmark

  Long (1990)
  
  \( V^* \) is best performing in several ways

• is growth optimal portfolio

  Kelly (1956)

  \[
  V^* = \max E(\log V_T)
  \]

  \[
  \pi_{t}^{1*} = \frac{\mu_t - \gamma - r_t}{\sigma_t^2} = \frac{\theta_t}{\sigma_t}
  \]

  \[
  \frac{dV_{t}^*}{V_{t}^*} = r_t \, dt + \theta_t (\theta_t \, dt + dW_t)
  \]

  Merton (1992)
• NP best performing portfolio

$$\limsup_{T \to \infty} \frac{1}{T} \log \left( \frac{V_T^*}{V_0^*} \right) \geq \limsup_{T \to \infty} \frac{1}{T} \log \left( \frac{V_T}{V_0} \right)$$

global well diversified index $\approx$ NP

Pl. (2005)
Figure 1: Discounted S&P500 total return index.
Figure 2: Logarithm of discounted S&P500.
• benchmarked price

\[ \hat{U}_t = \frac{U_t}{V_t^*} \]

\[ \hat{U} \text{ nonnegative} \implies \]

\[ \hat{U}_t \geq E_t (\hat{U}_s) \]

\[ t \leq s \]

**supermartingales** (no upward trend)

Pl. (2002)

no strong arbitrage
benchmarked securities that form martingales are called fair.

\[ \hat{U}_t = E_t (\hat{U}_s) \]

\[ t \leq s \]
• Law of the Minimal Price

Pl. (2008)

Fair prices are minimal prices

\( V \) nonnegative fair portfolio
\( V' \) nonnegative portfolio

\[ V_T = V'_T \]

supermartingale property
\[ V_t \leq V'_t \]

\( t \in [0, T], \)
Figure 3: Benchmarked savings bond and benchmarked fair zero coupon bond.
• real world pricing formula

for

\[ E_t \left( \frac{H_T}{V_{T*}} \right) < \infty \]

\[ U_H(t) = V_{t*} \ E_t \left( \frac{H_T}{V_{T*}} \right) \]

\[ t \in [0, T] \]

no risk neutral probability needs to exist
• actuarial pricing formula

when $H_T$ is independent of $V_T^*$

\[ U_H(t) = P(t, T) E_t(H_T) \]

zero coupon bond

\[ P(t, T) = V_t^* E_t \left( \frac{1}{V_T^*} \right) \]
• standard risk neutral pricing

Ross (1976), Harrison & Pliska (1983)

complete market

candidate Radon-Nikodym derivative

\[ \Lambda_t = \frac{dQ}{dP} \bigg|_{\mathcal{A}_t} = \frac{B_t V_0^*}{B_0 V_t^*} \]

supermartingale \[\implies\]

\[ 1 = \Lambda_0 \geq E_0 (\Lambda_T) \]
$$U_H(0) = E \left( \frac{V_0^*}{V_T^*} H_T \right)$$

$$= E \left( \Lambda_T \frac{B_0}{B_T} H_T \right) \leq \frac{E \left( \Lambda_T \frac{B_0}{B_T} H_T \right)}{E(\Lambda_T)}$$
Only in the special case when $\Lambda_T$ martingale

$\implies$ risk neutral pricing formula:

$$U_H(0) = E_Q \left( \frac{B_0}{B_T} H_T \right)$$

Ignores any real trend!
• Trends ignored also when using:

  stochastic discount factor, Cochrane (2001);

  deflator, Duffie (2001);

  pricing kernel, Constantinides (1992);

  state price density, Ingersoll (1987);

  NP as Long (1990)
• Example zero coupon bond

benchmarked fair zero coupon

\[ \hat{P}(t, T) = \frac{P(t, T)}{V_t^*} = E_t \left( \frac{1}{V_T^*} \right) \]

martingale (no trend)

for \( r_t \) - deterministic

\[ P(t, T) = V_t^* E_t \left( \frac{1}{V_T^*} \right) = \exp \left\{ - \int_t^T r_s \, ds \right\} E_t \left( \frac{\tilde{V}_t^*}{\tilde{V}_T^*} \right), \]

\[ \tilde{V}_t^* = \frac{V_t^*}{B_t} \quad \text{- discounted NP} \]
Figure 4: Benchmarked savings bond and benchmarked fair zero coupon bond.
downward trend reflects equity premium

\[ P(t, T) < \exp \left\{ - \int_{t}^{T} r_s \, ds \right\} = \frac{B_t}{B_T} \]
Figure 5: Radon-Nikodym derivative and total mass of putative risk neutral measure.
Figure 6: Savings bond, fair zero coupon bond and savings account.
Figure 7: Logarithms of savings bond and fair zero coupon bond.
discounted NP

\[ \bar{V}_t^* = \frac{V_t^*}{B_t} \]

\[ d\bar{V}_t^* = \bar{V}_t^* \theta_t (\theta_t \, dt + dW_t) \]

\[ = \alpha_t \, dt + \sqrt{\alpha_t} \, \bar{V}_t^* \, dW_t \]
• discounted NP drift

\[ \alpha_t := \bar{V}_t \theta_t^2 \]

\[ \theta_t = \sqrt{\frac{\alpha_t}{\bar{V}_t^*}} \]

volatility reflects leverage effect
• minimal market model
  MMM

  discounted NP drift
  assume
  \[ \alpha_t = \alpha_0 \exp\{\eta t\} \]
  \[ \alpha_0 > 0 \]
  net growth rate \( \eta > 0 \)
  \[ \implies \quad \text{MMM} \]
• normalized NP

\[ Y_t = \frac{\bar{V}_t^*}{\alpha_t} \]

\[ dY_t = (1 - \eta Y_t) \, dt + \sqrt{Y_t} \, dW_t \]

square root process of dimension four

Only one parameter!
• volatility of NP

\[
\theta_t = \frac{1}{\sqrt{Y_t}} = \sqrt{\frac{\alpha_t}{V_t^*}}
\]

leverage effect
- Zero coupon bond under the MMM

\( r_t \) - deterministic

\[
P(t, T) = \exp \left\{ - \int_t^T r_s \, ds \right\} \left( 1 - \exp \left\{ - \frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))} \right\} \right)
\]

Explicit formula!
Figure 8: Fraction invested in the savings account.
Figure 9: Benchmarked profit and loss.
• European put option under the MMM

Hulley, Miller & Pl. (2005)

\[ p(t, V_t^*, T, K, r) = V_t^* E_t \left( \frac{(K - V_{\tau}^*)^+}{V_{\tau}^*} \right) \]

\[ p(t, V_t^*, T, K, r) = -V_t^* \chi^2(d_1; 4, l_2) \]

\[ + Ke^{-r(T-t)} \left( \chi^2(d_1; 0, l_2) - \exp \{-l_2/2\} \right) \]
with

\[ d_1 = \frac{4\eta K \exp\{-r(T - t)\}}{B_t \alpha_t (\exp\{\eta(T - t)\} - 1)} \]

and

\[ l_2 = \frac{4\eta V^*_t}{B_t \alpha_t (\exp\{\eta(T - t)\} - 1)} \]
\( \chi^2(x; n, l) \) non-central chi-square distribution function

\( n \geq 0 \) degrees of freedom

non-centrality parameter \( l > 0 \)

\[
\chi^2(x; n, l) = \sum_{k=0}^{\infty} \frac{\exp \left\{ - \frac{l}{2} \right\} \left( \frac{l}{2} \right)^k}{k!} \left( 1 - \frac{\Gamma \left( \frac{x}{2}; \frac{n+2k}{2} \right)}{\Gamma \left( \frac{n+2k}{2} \right)} \right)
\]
• putative risk neutral price

\[ \tilde{p}(t, V_t^*, T, K, r) = p(t, V_t^*, T, K, r) + Ke^{-r(T-t)} \exp \left\{ -\frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))} \right\} \]

risk neutral is overpricing

for \( \bar{V}_t^* \to 0 \quad \Rightarrow \quad \tilde{p}(t, V_t^*, T, K, r) \to 0 \)

and

\[ \tilde{p}(t, V_t^*, T, K, r) \to Ke^{-r(T-t)} > 0 \]

risk neutral ignores trends
Figure 10: Logarithms of savings bond times $K$, risk neutral put and fair put.
Guaranteed Minimum Death Benefit (GMDB)

- payout to the policyholder

\[ \max(e^{g\tau}V_0, V_\tau) \]

- time of death \( \tau \)

- \( g \geq 0 \) is the guaranteed instantaneous growth rate

- \( V_0 \) is the initial account value

- \( V_\tau \) is the unit value of the policyholder’s account at time of death \( \tau \)
• insurance charges $\xi \geq 0$

$\implies$ policyholder’s unit value

$$V_t = e^{-\xi t} V^*_t$$

$\implies$

$$\max(e^{g\tau} V_0, V_\tau) = \max(e^{g\tau} V_0, e^{-\xi \tau} V^*_\tau)$$

$$= e^{-\xi \tau} \max(e^{(g+\xi)\tau} V_0, V^*_\tau)$$
insurance company invests the entire fund value $V$ in the NP $V^*$

$\implies$ payoff

$$H_T = GMDB_T = e^{-\xi T} \left[ (e^{(g+\xi)T}V_0^* - V_T^*)^+ + V_T^* \right]$$
fair value $GMDB_0$ of the total claim

real world pricing formula

$$GMDB_0 = V_0^* E \left( \frac{GMDB_T}{V_T^*} \right)$$

$$= e^{-\xi T} \left( p(0, V_0^*, T, e^{(g+\xi)T}V_0^*, r) + V_0^* \right)$$
Figure 11: Present value of the GMDB under the real world pricing formula (left), the risk neutral pricing formula (middle) and the Black Scholes formula (right) for $\eta = 0.05$, $\alpha_0 = 0.05$, $r = 0.05$, $\xi = 0.01$ and $Y_0 = 20$. 
• lifetime $\tau$ is stochastic

$$GMDB_0 = V_0^* \ E \left( E \left( \frac{GMDB_\tau}{V_\tau^*} \mid \mathcal{F}_\tau \right) \right)$$

$$GMDB_0 = \int_0^T \left( p(0, V_0^*, t, e^{(g+\xi)t} V_0^*, r) + V_0^* \right) e^{-\xi t} f_\tau(t) \ dt$$

$f_\tau(\cdot)$ - mortality density
Figure 12: German mortality data
rational investors will lapse when embedded put-option out of the money
GMDBs have stochastic maturity
Titanic options, Milevsky & Posner (2001)

- roll-up GMDB payoff

\[ H_t = \begin{cases} 
(1 - \beta_t)V_t^*, & \text{if lapsed at time } t, \\
\max(e^{\gamma \tau} V_0, V_{\tau}^*), & \text{if death occurs at time } t = \tau 
\end{cases} \]

surrender charge \( \beta_t \)

\[ \beta_t = \begin{cases} 
(8 - \lfloor t \rfloor)\%, & t \leq 7, \\
0, & t > 7
\end{cases} \]

no credit risk

no accumulation phase
Figure 13: Value of the GMDB under the MMM for male and female policyholders aged $\mathbf{a}$, assuming an irrational lapsation of $l = 1\%$. 

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References


