

# Economic Pricing of Mortality-Linked Securities: A Tâtonnement Approach

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# Outline

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Tâtonnement Approach – Multi-Period

An Illustrative Mortality-Linked Security

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# Motivation

## Mortality-Linked Risks

- ▶ Life insurers: mortality deterioration
- ▶ Pension plan sponsors: mortality improvement

## Mortality-linked securities

- ▶ Hedging against unexpected change in mortality rates
- ▶ Eg. Swiss Re mortality bond, BNP/EIB longevity bond

# How to Price Mortality-Linked Securities?

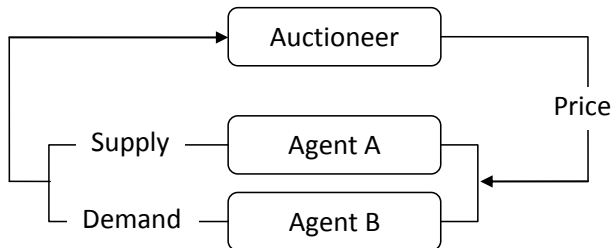
## No arbitrage approach

- ▶ E.g. risk-adjusted two-factor mortality model, Wang transform, canonical valuation
- ▶ Disadvantages
  - ▶ takes actual prices of other products as given
  - ▶ cannot identify a unique risk-neutral measure except canonical valuation

## Our approach

- ▶ does not need market prices of other products
- ▶ provides a unique price
- ▶ requires more model structure

## Our Idea

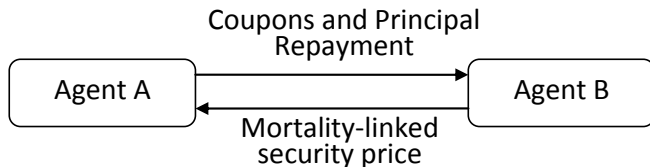


- ▶ The auctioneer adjusts the price to match supply and demand
- ▶ Price adjustment stops when supply equals to demand

## Questions

- ▶ How to determine supply and demand?
- ▶ How to adjust price?

## A One-Period Mortality-Linked Security



- ▶  $q_1$ : a mortality index
- ▶ Agent A (the hedger) issues a mortality-linked security.
- ▶ At time 1, Agent A obligates to pay:
  - ▶ life insurance/annuity benefits,  $f(q_1)$
  - ▶ payoff from the mortality-linked security,  $g(q_1)$  per unit
- ▶ Agent B (the investor) receives the payoff  $g(q_1)$  per unit at time 1

# Notation and Assumptions

## Notation

- ▶  $\theta^A$  and  $\theta^B$ : supply and demand from Agents A and B
- ▶  $\omega^A$  and  $\omega^B$ : the initial wealths of Agents A and B
- ▶  $r$ : continuously compounding risk-free interest rate
- ▶  $U^A$  and  $U^B$ : utility functions for Agents A and B

## Assumptions

- ▶ Two investment choices
  - ▶ the mortality-linked security
  - ▶ a bank account paying an interest rate  $r$
- ▶ Allow the agents to borrow money at an interest rate  $r$

## How to Determine Supply and Demand?

### Maximizing expected terminal utilities

$$\text{Agent A: } \sup_{\theta^A} \mathbb{E}[U^A((\omega^A + \theta^A P)e^r - \theta^A g(q_1) - f(q_1)))]$$

$$\text{Agent B: } \sup_{\theta^B} \mathbb{E}[U^B((\omega^B - \theta^B P)e^r + \theta^B g(q_1)))]$$

## How to Adjust the Price?

$$P_{k+1} = P_k + h_k(\theta^B - \theta^A), \quad k = 1, 2, \dots$$

- ▶  $P_0$  is the initial guess of the price
- ▶ If  $\theta^B > \theta^A$ , then  $h_k(\theta^B - \theta^A) > 0$  and the price is raised
- ▶ If  $\theta^B < \theta^A$ , then  $h_k(\theta^B - \theta^A) < 0$  and the price is lowered
- ▶ We set  $h_k(\theta^B - \theta^A) = \gamma |P_k| (\theta^B - \theta^A)$ , where  $\gamma > 0$

## Algorithm 1

1. Guess a price  $P_0$ , and set  $k = 0$ ;
2. Determine  $\theta^B$  and  $\theta^A$  under  $P_k$ ;
3. If  $\theta^A = \theta^B$ , stop. Otherwise, adjust the price and  $k = k + 1$ ;
4. Repeat Steps 2 and 3.

## A Multi-Period Setup

### More notation

- ▶  $T$ : maturity time
- ▶  $Q_t = (q_1, \dots, q_t)$ , where  $q_t$  is the mortality index over the period of  $t - 1$  to  $t$
- ▶ Payments at  $t = 1, \dots, T$ 
  - ▶ Life insurance/annuity benefits:  $f_t(Q_t)$
  - ▶ Payouts from the mortality-linked security:  $g_t(Q_t)$  per unit

# No Trade after Time 0

## Wealth Processes

$$\begin{aligned} \text{Agent A} \quad W_0^A &= \omega^A \\ W_1^A &= (W_0^A + \theta^A P)e^r - \theta^A g_1(Q_1) - f_1(Q_1) \\ W_t^A &= W_{t-1}^A e^r - \theta^A g_t(Q_t) - f_t(Q_t) \end{aligned}$$

$$\begin{aligned} \text{Agent B} \quad W_0^B &= \omega^B \\ W_1^B &= (W_0^B - \theta^B P)e^r + \theta^B g_1(Q_1) \\ W_t^B &= W_{t-1}^B e^r + \theta^B g_t(Q_t) \end{aligned}$$

where  $t = 2, 3, \dots, T$

# No Trade after Time 0

## Price Adjustment

1. Guess a price  $P_0$ , and set  $k = 0$ .
2. Determine supply and demand by

$$\theta^A = \underset{\theta^A}{\operatorname{argsup}} \mathbb{E}[U^A(W_T^A)]$$

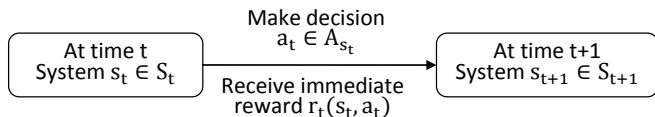
$$\theta^B = \underset{\theta^B}{\operatorname{argsup}} \mathbb{E}[U^B(W_T^B)]$$

3. If  $\theta^A = \theta^B$ , stop. Otherwise, adjust price by  $P_{k+1} = P_k + h_k(\theta^B - \theta^A)$ .
4. Repeat step 2 and 3.

## Allowing Trades after Time 0

- ▶ Determine supply and demand at  $t = 0, 1, \dots, T - 1$
- ▶ Make a sequence of decisions, such that the expected terminal utility is maximized
- ▶ Apply theories in sequential decision process
  - ▶ System: the mortality-linked security market
  - ▶ State variable: mortality index, trading information
  - ▶ Current state: known
  - ▶ Future states: uncertain

## A Discrete Markov Decision Process



- ▶  $s_t$ : system state at  $t$
- ▶  $a_t$ : action at  $t$ , depending on  $s_t$
- ▶  $r_t(s_t, a_t)$ : reward at  $t$
- ▶  $A_{s_t}$ : set of allowable actions in state  $s_t$
- ▶ Goal: Maximizing the expected total reward

$$\mathbb{E}[\sum_{i=0}^{T-1} r_i(s_i, a_i) + r_T(s_T)]$$

## Optimality equations

$$v_t(s_t) = \begin{cases} r_T(s_T) & t = T \\ \sup_{a \in A_{s_t}} \{r_t(s_t, a) + \mathbb{E}[u_{t+1}(s_{t+1}) | s_t, a]\} & \text{otherwise} \end{cases}$$

- ▶ Optimal actions:  $[a_0, a_1, \dots, a_{T-1}]$ , where
 
$$a_t = \operatorname{argsup}_{a \in A_{s_t}} \{r_t(s_t, a) + \mathbb{E}[v_{t+1}(s_{t+1}) | s_t, a]\}$$
- ▶ In our problem,
  - ▶  $r_T(s_T) = U(W_T)$
  - ▶  $r_t(s_t, a_t) = 0$ , for  $t = 0, 1, \dots, T - 1$

## The Lee-Carter Model (Lee and Carter, 1992)

$$\begin{aligned}\ln(m_{x,t}) &= \beta_x^{(0)} + \beta_x^{(1)} \kappa_t + \epsilon_{x,t} \\ \kappa_{t+1} &= \kappa_t + \mu + \sigma Z_{t+1}\end{aligned}$$

- ▶  $m_{x,t}$ : central death rate at age  $x$  and in year  $t$
- ▶  $\beta_x^{(0)}$ : average level of mortality (in log scale) over time
- ▶  $\beta_x^{(1)}$ : age-specific sensitivity to the time-varying factor
- ▶  $\kappa_t$ : the time-varying factor
- ▶  $\epsilon_{x,t}$ : the error term

## The Lee-Carter Model with Jumps (Chen and Cox, 2009)

$$\kappa_{t+1} = \begin{cases} \kappa_t + \mu - \rho m + \sigma Z_{t+1}, & \text{if } N_{t+1} = 0, \\ \kappa_t + \mu - \rho m + \sigma Z_{t+1} + Y_{t+1}, & \text{if } N_{t+1} = 1, \end{cases}$$
$$N_t = \begin{cases} 1, & \text{with probability } \rho, \\ 0, & \text{with probability } 1 - \rho, \end{cases}$$

- ▶  $Y_t \sim N(m, s^2)$ : jump severity variable at time  $t$
- ▶  $\mu$  and  $\sigma$ : constants
- ▶  $Z_t \sim N(0, 1)$ : independent of both  $Y_t$  and  $N_t$ .
- ▶  $N_t$ : number of jumps occurring in year  $t$

# The Lee-Carter Model with Jumps

## Model calibration

- ▶ Chen and Cox (2009) fitted the model to US mortality data from National Center for Health Statistics (NCHS)

Parameter	Estimate	Parameter	Estimate
$\mu$	-0.2137	$\sigma$	0.3733
$m$	0.8393	$s$	1.4316
$\rho$	0.0436	log-likelihood	63.49

## Discretization of the State Space (Amin, 1993)

- ▶  $\kappa_t - \kappa_0 = t(\mu - pm) + \sigma \sum_{i=1}^t Z_i + \sum_{i=1}^t N_i Y_i$
- ▶  $u_t = \kappa_t - \kappa_0$
- ▶  $\alpha = \mu - pm$

Time = 0	Time = 1	...	Time = t
	$\vdots$	$\dots$	$\vdots$
	$u_1(3) = \alpha + 3\sigma$	$\dots$	$u_t(3) = \alpha t + 3\sigma$
	$u_1(2) = \alpha + 2\sigma$	$\dots$	$u_t(2) = \alpha t + 2\sigma$
	$u_1(1) = \alpha + \sigma$	$\dots$	$u_t(1) = \alpha t + \sigma$
$u_0(0) = 0$	$u_1(0) = \alpha$	$\dots$	$u_t(0) = \alpha t$
	$u_1(-1) = \alpha - \sigma$	$\dots$	$u_t(-1) = \alpha t - \sigma$
	$u_1(-2) = \alpha - 2\sigma$	$\dots$	$u_t(-2) = \alpha t - 2\sigma$
	$u_1(-3) = \alpha - 3\sigma$	$\dots$	$u_t(-3) = \alpha t - 3\sigma$
	$\vdots$	$\dots$	$\vdots$

## Discretization of State Space

### Local changes and jumps

	⋮	Jump
	$u_{t+1}(i+2)$	Jump
	$u_{t+1}(i+1)$	Local change
$u_t(i)$	$u_{t+1}(i)$	Jump
	$u_{t+1}(i-1)$	Local change
	$u_{t+1}(i-2)$	Jump
	⋮	

### Tree truncation

- ▶ Truncate the number of states at each time step to  $2M + 1$ .
- ▶ Assign transition probabilities according to Amin (1993).

# An Illustrative Mortality-Linked Security

## Assumptions

- ▶  $T = 3$  years
- ▶ Face value = \$1
- ▶  $r = 3\%$
- ▶ Annual coupon rate =  $r + 1.5\%$
- ▶  $q_t = m_{65+t, 2003+t}$
- ▶ Life insurance benefits  $f_t(Q_t) = 50q_t$
- ▶ Principal Repayment =  $\max\left(1 - \sum_{t=1}^3 \text{loss}_t\right)$
- ▶  $\text{loss}_t = \frac{\max(q_t - 1.1q_0, 0) - \max(q_t - 1.3q_0, 0)}{0.2q_0}$

## An Illustrative Mortality-Linked Security

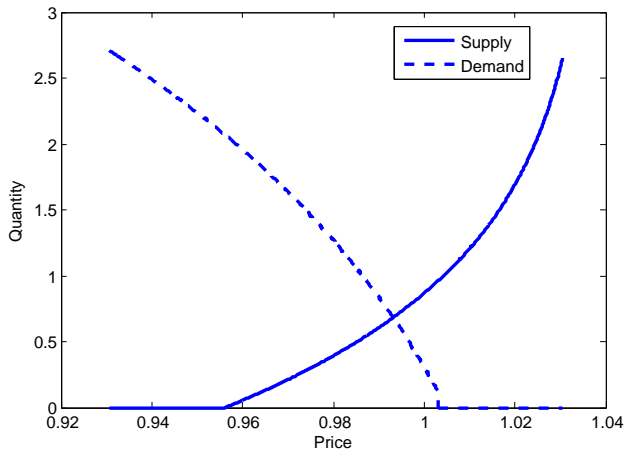
### Assumptions

- ▶ Exponential utility function  $U(x) = 1 - e^{-kx}$
- ▶ Absolute risk aversion  $k^A = 1.0$ ,  $k^B = 0.5$

### Results (No Trade after Time 0)

- ▶ Estimated price of the mortality-linked security is **\$0.9936**.
- ▶ Optimal quantity of the security traded is **0.6116 units**.
- ▶ Implied risk premium is **36 basis points** per annum.

## Supply and Demand Curves

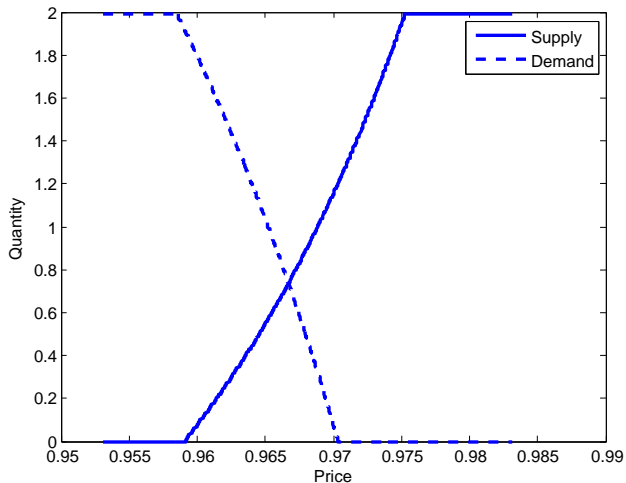


## Results (Allowing Trades after Time 0)

►  $\theta_t \in \{0, 0.01, 0.02, \dots, 1.98, 1.99\}$

$M$	3	4	5	6	7	8
Price	0.9763	0.9695	0.9674	0.9669	0.9667	0.9667
Quantity	0.93	0.70	0.72	0.74	0.75	0.75

## Supply and Demand Curves



## The Risk-Neutral Measure in Chen and Cox (2009)

$$\tilde{\kappa}_{t+1} = \begin{cases} \tilde{\kappa}_t + \mu - pm + \sigma \tilde{Z}_{t+1} & \text{if } \tilde{N}_{t+1} = 0 \\ \tilde{\kappa}_t + \mu - pm + \sigma \tilde{Z}_{t+1} + \tilde{Y}_{t+1} & \text{if } \tilde{N}_{t+1} = 1 \end{cases}$$

$$\tilde{N}_t = \begin{cases} 1 & \text{with probability } \tilde{p} \\ 0 & \text{with probability } 1 - \tilde{p} \end{cases}$$

where

$$\begin{aligned} \tilde{Z}_t &\sim N(\lambda_1, 1) \\ \tilde{Y}_t &\sim N(m + \lambda_2 s, s^2) \\ \hat{p} &= 1 - \Phi[\Phi^{-1}(1 - p) - \lambda_3] \end{aligned}$$

$[\lambda_1, \lambda_2, \lambda_3]$  are solved by **Actual price = Risk-neutral price**

## Comparing the Results

### Chen and Cox's results

$\lambda_1$	4.6408	0	0.8072
$\lambda_2$	0	2.0006	0.8072
$\lambda_3$	0	0	0.8072
Price	\$0.1511	\$0.9728	\$0.8053

### Our result

- ▶ Price = \$0.9936

## Sensitivity Test – Initial Wealth

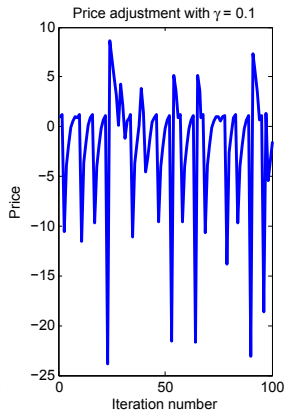
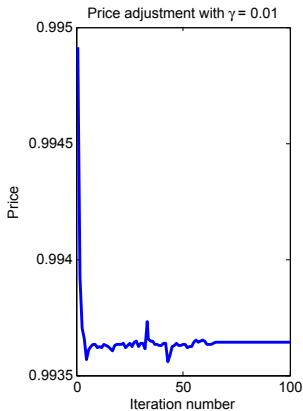
### Proposition

*If exponential utility functions are assumed,  $\omega_A$  and  $\omega_B$  have no effect on the estimated price.*

## Sensitivity Test – Risk Aversion Parameters

$k^A$	$k^B$	Price	Quantity	$k^A$	$k^B$	Price	Quantity
1	0.25	0.9981	0.7327	0.5	0.5	0.9973	0.4172
1	0.5	0.9936	0.6116	1	0.5	0.9936	0.6116
1	1	0.9837	0.4590	2	0.5	0.9868	0.9383
1	2	0.9799	0.3061	4	0.5	0.9697	1.6052
2	4	0.9062	0.3962	4	2	0.7885	1.2177

# Sensitivity Test – $P_{k+1} = P_k + \gamma |P_k| (\theta^B - \theta^A)$



## Sensitivity Test – the Risk-free Interest Rate

$r$	0.03	0.04	0.05	0.06
Price	\$0.9936	\$0.9931	\$0.9924	\$0.9913
Units traded	0.6116	0.6175	0.6236	0.6297

## Conclusion

- ▶ No trade after time 0
  - ▶ simple and straightforward to implement
  - ▶ suitable for an illiquid mortality-linked security market
- ▶ Allowing trades after time 0
  - ▶ permitting the counterparties to unwind their positions
  - ▶ requiring more computational resources
- ▶ The method does not take actual market prices as given
- ▶ It is ideal for pricing bespoke deals
- ▶ It may be used to construct hedging strategy

## Future Work

- ▶ Examine conditions for the existence and uniqueness of an equilibrium price.
- ▶ Develop a more computationally efficient method for the model that permits trades after time 0.